A CATEGORICAL INVARIANT OF FLOW EQUIVALENCE OF SHIFTS

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Abstract. The Karoubi envelope of a shift is defined as the Karoubi envelope of the syntactic semigroup of the language of blocks of the shift. We prove that, up to natural equivalence of categories, the Karoubi envelope of a shift is invariant under flow equivalence. More precisely, we show that the action of the Karoubi envelope on the Krieger cover of the shift is a flow invariant. An analogous result concerning the Fischer cover of a synchronizing shift is also obtained.

From these main results, several flow equivalence invariants — some new and some old — are obtained. Another application concerns the classification of Markov-Dyck shifts: it is shown that, under mild conditions, two graphs define flow equivalent Markov-Dyck shifts if and only if they are isomorphic.

Contents

1. Introduction 2
2. The Karoubi envelope of a semigroup 4
3. Symbolic dynamics 7
4. Statement of the main results 14
5. The proper comunication graph 16
6. The labeled preordered set of the $\mathcal{D}$-classes of $S(\mathscr{X})$ 18
7. Classes of sofic shifts defined by pseudovarieties of semigroups 23
8. Subsynchronizing subshifts of a sofic shift 25
9. Markov-Dyck shifts 28

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10. Eventual conjugacy 33
11. Preliminaries for the proof of Theorem 4.4 35
12. Proof of Theorem 4.4 40
13. Proof of Theorem 4.6 49
Appendix A. A remark on ordered semigroups 50
Appendix B. Symbol expansion and subsynchronizing subshifts 51
References 51

1. Introduction

Two discrete-time dynamical systems are *flow equivalent* if their suspension flows are equivalent. For symbolic dynamical systems, Parry and Sullivan characterized flow equivalence as the equivalence relation between shifts generated by conjugacy and a non-symmetric relation which at present is called *symbol expansion* [61]; see also [52, Section 13.7] and [3, 5]. Special attention has been given to the classification of shifts of finite type up to flow equivalence; in that context, complete and decidable algebraic invariants were obtained, first for the irreducible case [27], then for the reducible case [13, 35]. However, as pointed out in [3], only small progress have been made in the strictly sofic case, where one finds few useful flow equivalence invariants, even for irreducible shifts.

The role of the syntactic semigroup of the language of finite blocks of a shift $\mathscr{X}$, which we call the *syntactic semigroup of* $\mathscr{X}$, has been considered in the literature [6–8, 17, 18, 39, 40] essentially in the context of (strictly) sofic shifts. In [21], one finds a characterization of the abstract semigroups which are the syntactic semigroup of a sofic shift.

For a semigroup S, let $\mathbb{K}(S)$ be the Karoubi envelope (also known as Cauchy completion or idempotent splitting) of S. It is a certain small category that plays a crucial role in finite semigroup theory thanks in part to the Delay Theorem of Tilson (see [71], where $\mathbb{K}(S)$ is denoted by S_E).

For a shift $\mathscr{X}$, let $S(\mathscr{X})$ be its syntactic semigroup. In this paper we prove that the Karoubi envelope $\mathbb{K}(S(\mathscr{X}))$, more briefly denoted $\mathbb{K}(\mathscr{X})$, is a flow equivalence invariant of $\mathscr{X}$, up to equivalence of categories. This says in a sense to be made more precise that $\mathscr{X}$ determines $S(\mathscr{X})$ up to Morita equivalence.

The category $\mathbb{K}(\mathscr{X})$ is of little use when dealing with shifts of finite type. Indeed, a shift of finite type is conjugate with an edge shift, and it is easy to see that if $\mathscr{X}$ is an irreducible edge shift then $S(\mathscr{X})$ is isomorphic to a Brandt semigroup B_n for some n [19, Remark 2.23]. Unfortunately, all finite Brandt semigroups have equivalent Karoubi envelopes. However, for other classes, including strictly sofic shifts, we do obtain interesting results.

We also investigate the actions of the category $\mathbb{K}(\mathscr{X})$ on the Krieger cover of $\mathscr{X}$ and, if $\mathscr{X}$ is synchronizing, on its Fischer cover. We show that these actions are invariant under flow equivalence. This result is applied in

Section 5 to obtain a new proof of the invariance under flow equivalence of the proper communication graph of a sofic shift (a result from [3]), and in the process we obtain a generalization to arbitrary shifts.

In Section 6, we provide examples of pairs of almost finite type shifts $\mathscr{X}$ and $\mathscr{Y}$ such that $\mathbb{K}(\mathscr{X})$ and $\mathbb{K}(\mathscr{Y})$ are not equivalent, whereas other flow equivalence invariants fail to separate them. In the first of these examples we use a labeled poset, hailing from Green's relations on $S(\mathscr{X})$, which was shown in [17] to be a conjugacy invariant of sofic shifts. This is a more refined version of a previously invariant considered in [7].

As a consequence of the flow invariance of $\mathbb{K}(\mathscr{X})$ (up to equivalence of categories), we recover and extend the invariant from [17] to the context of flow equivalence, as well to non-sofic shifts. We mention that the proofs in [7] are based on Nasu's Classification Theorem for sofic shifts [59], while those in [17] rely on using some mappings between free semigroups naturally defined from sliding block codes of shifts. This second approach, properly adapted from [17] to cover the case of non-sofic shifts, is again used in the three final technical sections of the paper.

In Sections 7 and 8, using the Karoubi envelope of $S(\mathscr{X})$, we recover other conjugacy invariants of sofic shifts, showing that they are actually flow equivalence invariants. In Section 7 we deal with the invariants of [18], in particular deducing again that the class of almost finite type shifts is stable under flow equivalence (Theorem 7.5), a result from [28]. In Section 8 we conclude that the poset of subsynchronizing subshifts of a sofic shift, studied in [40], is invariant under flow equivalence.

The Karoubi envelope is applied in Section 9 to classify up to flow equivalence a class of non-sofic shifts, the Markov-Dyck shifts of Krieger and Matsumoto [48], thus reproving and generalizing the classification of Dyck shifts, first obtained in [58].

The basic outline of the paper is as follows. We begin with two preliminaries sections. The first contains some aspects of category theory and semigroup theory that we shall require in order to extract flow equivalence invariants from the category $\mathbb{K}(\mathscr{X})$. The second preliminary section is about symbolic dynamics. Section 4 states our main results. In Sections 5 to 9, consequences of our main results are explored, as well as relations with previous work, from the viewpoint of the classification of shifts up to flow equivalence. Although the principal motivation in this paper is the study of flow equivalence, we also deduce in Section 10 that several of our flow invariants are in addition eventual conjugacy invariants in the case of sofic shifts. Finally, we give the proofs of our main results in the last three sections.

There are also two short appendices. The first one contains remarks about the flow invariance of $\mathbb{K}(\mathscr{X})$, up to natural equivalence, when $S(\mathscr{X})$ is viewed as an ordered semigroup. The second appendix completes an argument in the proof of Theorem 8.8, using technical tools from Section 12.

2. The Karoubi envelope of a semigroup

2.1. **Categorical preliminaries.** The reader is referred to [10–12, 54] for basic notions from category theory. A category C is *small* if its objects and arrows form a set. We recall that two categories C and D are *equivalent* if there are functors $F\colon C \longrightarrow D$ and $G\colon D \longrightarrow C$ such that $FG \cong 1_D$ and $GF \cong 1_C$, where the symbol $\cong$ stands for isomorphism of functors. A functor $F\colon C \longrightarrow D$ between small categories is an equivalence (i.e., there exists such a G) if and only if it is fully faithful and essentially surjective. *Fully faithful* means bijective on hom-sets, whereas *essentially surjective* means that every object of D is isomorphic to an object of $F(C)$. The former is in accordance with the usual terminology for functors which are injective on hom-sets (the *faithful* functors) and for those surjective on hom-sets (the *full* functors).

2.2. **The Karoubi envelope and Morita equivalence of semigroups.** An important notion in this paper is the *Karoubi envelope* $\mathbb{K}(S)$ (also known as Cauchy completion or idempotent splitting) of a semigroup S. It is a small category whose object set is the set $E(S)$ of idempotents of S. Morphisms in $\mathbb{K}(S)$ from f to e are represented by arrows $e \longleftarrow f$. It will be explained in the following paragraph why we prefer to put the source of an arrow of $\mathbb{K}(S)$ on the right. A morphism $e \longleftarrow f$ is a triple (e, s, f) where $s \in eSf$. Note that $s \in eSf$ if and only if $s = esf$, because e and f are idempotents. Composition of morphisms is given by

$$(e, s, f)(f, t, g) = (e, st, g).$$

The identity at e is (e, e, e). It is easy to show that idempotents e, f are isomorphic in $\mathbb{K}(S)$ if and only if there exist $x, x' \in S$ such that $xx'x = x$, $x'xx' = x'$, $x'x = e$ and $xx' = f$. In semigroup terms (which will be explained in more detail in Subsection 2.4), this says that e, f are $\mathcal{D}$-equivalent [68], whereas in analytic terms this corresponds to von Neumann-Murray equivalence.

In Semigroup Theory, a triple (e, s, f) such that $s \in eSf$ is normally viewed as a morphism with domain e and co-domain f, and the composition of morphisms is taken in the direction opposite to that usually followed in Category Theory. This is because one tends to read a semigroup product from left to right. Accordingly to this alternate convention, we still have $(e, s, f)(f, t, g) = (e, st, g)$. In this paper, we adopt the Category Theory convention for composition. However, we do not want to deviate from the Semigroup Theory notation, and that is why we represent graphically a morphism (e, s, f) as an arrow $e \longleftarrow f$ with source on the right. All other categories will be treated as usual with arrows drawn from left to right.

If e is an idempotent of the semigroup S, then eSe is a a monoid with identity e, which is called the *local monoid of S at e*. The *local monoid of a category C at an object c* is the endomorphism monoid of c in C. The local monoids of S correspond to the local monoids of $\mathbb{K}(S)$, more precisely, eSe and $\mathbb{K}(S)(e, e)$ are isomorphic for every $e \in E(S)$.

An element s of a semigroup S has *local units e and f*, where e and f are idempotents of S, if $s = esf$. The set $LU(S) = E(S)SE(S)$ of elements of S with local units is a subsemigroup of S. If $LU(S) = S$, then we say that S has *local units*. In general, $LU(S)$ is the largest subsemigroup of S which has local units (it may be empty). Clearly $\mathbb{K}(S) = \mathbb{K}(LU(S))$ and so the Karoubi envelope does not distinguish between these two semigroups. Talwar defined in [70] a notion of Morita equivalence of semigroups with local units in terms of equivalence of certain categories of actions. It was shown in [29, 51] that semigroups S and T with local units are Morita equivalent if and only if $\mathbb{K}(S)$ and $\mathbb{K}(T)$ are equivalent categories. Thus we shall say that semigroups S and T are *Morita equivalent up to local units* if $\mathbb{K}(S)$ and $\mathbb{K}(T)$ are equivalent categories, or in other words if $LU(S)$ is Morita equivalent to $LU(T)$. In this paper, we will show that flow equivalent shifts have syntactic semigroups that are Morita equivalent up to local units.

2.3. **Categories with zero.** A *semigroup with zero* is a semigroup S with an element 0 such that $0s = 0 = s0$ for all $s \in S$. The element 0 is unique.

A *category with zero* is a category C enriched over the monoidal category of pointed sets. What this means concretely is that, for all objects c, d of C, there is a zero morphism $0_{c,d} \in C(c, d)$ satisfying the following property. For all $f\colon c' \longrightarrow c$ and $g\colon d \longrightarrow d'$ one has

$$0_{c,d}f = 0_{c',d} \quad \text{and} \quad g0_{c,d} = 0_{c,d'}.$$

For ease of notation, we put $0_{c,c} = 0_c$. For instance, every abelian category is a category with 0. The most important example for us is the case where S is a semigroup with zero and C is the Karoubi envelope $\mathbb{K}(S)$. Then $0_{e,f} = (f, 0, e)$ is the zero morphism of $\mathbb{K}(S)(e, f)$. An object c of a category with zero is said to be *trivial* if $1_c = 0_c$. Notice then that the only morphisms into and out of a trivial object are the zero morphisms.

Observe that if $f \in C(c, d)$, then $0_d f = 0_{c,d}$. Thus the zero morphisms are determined by the zero morphisms of local monoids. Since a zero element of a monoid is unique, it follows that the zero morphisms of a category with zero are uniquely determined. From now on, we will drop the subscripts on zero morphisms when convenient. Note that if F is a full functor between categories with zero, then $F(0) = 0$. This enables us to register the following remark, for later reference.

Remark 2.1. If F is a fully faithful functor between categories with zero then $F(x) = 0$ if and only if $x = 0$.

2.4. **Green's relations and the Karoubi envelope.** Throughout this paper we shall use basic notions from semigroup theory that can be found in standard texts [1, 15, 31, 49, 68]. Green's (equivalence) relations, which we next recall, are among them. The relation $\mathcal{J}$ is defined on a semigroup S by putting $s \mathrel{\mathcal{J}} t$ if s and t generate the same two-sided principal ideal, that is, if $S^1sS^1 = S^1tS^1$, where S^1 denotes the monoid obtained from S adjoining an identity. We write $s \leq_{\mathcal{J}} t$ if $S^1sS^1 \subseteq S^1tS^1$, that is, if t is a

factor of s. The relation $\leq_{\mathcal{J}}$ thus defined is a preorder. Similarly, one defines the $\mathcal{R}$- and $\mathcal{L}$-relations, as well as the preorders $\leq_{\mathcal{R}}$ and $\leq_{\mathcal{L}}$, by replacing two-sided ideals with right (respectively, left) ideals. The intersection of the equivalence relations $\mathcal{R}$ and $\mathcal{L}$ is denoted by $\mathcal{H}$. The maximal subgroups of S are the $\mathcal{H}$-classes containing idempotents. Finally, Green's relation $\mathcal{D}$ is the least equivalence relation containing $\mathcal{R}$ and $\mathcal{L}$; it is known that $\mathcal{D} = \mathcal{L} \circ \mathcal{R} = \mathcal{R} \circ \mathcal{L}$. In the class of *stable* semigroups, which includes finite and compact semigroups, one has $\mathcal{D} = \mathcal{J}$ [68, Appendix A].

An element s of a semigroup S is *regular* if $s = sxs$ for some $x \in S$. If $\mathcal{K}$ is one of Green's relation, then we say that a $\mathcal{K}$-class is *regular* if its elements are regular. An element is regular if and only if its $\mathcal{D}$-class is regular.

Let H be an $\mathcal{H}$-class of S. The set $\{x \in S^1 \colon xH \subseteq H\}$ is a submonoid of S^1, called the *left stabilizer* of H. The quotient of T by its left action on H is a group known as the *Schützenberger group of H*. Exchanging right and left, one obtains an isomorphic group. If H is a group (which occurs if and only if it contains an idempotent), then it is isomorphic to its Schützenberger group. Two $\mathcal{H}$-classes contained in the same $\mathcal{D}$-class have isomorphic Schützenberger groups, hence the expression *Schützenberger group of a $\mathcal{D}$-class* is meaningful. See [16] for details.

Remark 2.2. Let S be a semigroup. If $\mathcal{K}$ is one of Green's relations $\mathcal{J}$, $\mathcal{R}$ or $\mathcal{L}$, then $s \leq_{\mathcal{K}} t$ in $LU(S)$ if and only if $s \leq_{\mathcal{K}} t$ in S, for all $s, t \in LU(S)$, and the Schützenberger group of $s \in LU(S)$ is the same in S as in $LU(S)$.

The following lemma concerning the Karoubi envelope $\mathbb{K}(S)$ of a semigroup is well known and easy to prove [51, 68].

Lemma 2.3. *Two objects e and f of $\mathbb{K}(S)$ are isomorphic if and only if e and f are $\mathcal{D}$-equivalent in S. Moreover, the automorphism group of e is isomorphic to the Schützenberger group of the $\mathcal{D}$-class of e.*

2.5. **The action of $\mathbb{K}(S)$.** Recall that a (right deterministic) *action of a semigroup S on a set Q* is a function $\mu\colon Q \times S \longrightarrow S$, with notation $\mu(q, s) = q \cdot s = qs$, such that $q \cdot (st) = (q \cdot s) \cdot t$. The action defines a function φ from S to the set Q^Q of transformation of Q, given by $\varphi(s)(q) = q \cdot s$. The action is *faithful* if φ is injective.

A (right deterministic) *action* of a small category C on a set Q is a contravariant functor $\mathbb{A}\colon C \longrightarrow \mathsf{Set}$ such that $\mathbb{A}(c)$ is a subset of Q for every object c of C, i.e., a *presheaf* on C taking values in subsets of Q [53]. If $s\colon c \longrightarrow d$ is a morphism of C, we may use the notation $q \cdot s$ for $\mathbb{A}(s)(q)$, where $q \in \mathbb{A}(d)$, and the notation $\mathbb{A}(d) \cdot s$ for the image of the function $\mathbb{A}(s)\colon \mathbb{A}(d) \longrightarrow \mathbb{A}(c)$. The notations $q \cdot s$ and $\mathbb{A}(d) \cdot s$ may be simplified to qs and $\mathbb{A}(d)s$.

Definition 2.4 (Equivalent actions)**.** Consider two actions $\mathbb{A}\colon C \longrightarrow \mathsf{Set}$ and $\mathbb{A}'\colon D \longrightarrow \mathsf{Set}$. We say that $\mathbb{A}$ and $\mathbb{A}'$ are *equivalent*, and write $\mathbb{A} \sim \mathbb{A}'$, if there is an equivalence $F\colon C \longrightarrow D$ such that $\mathbb{A}$ and $\mathbb{A}' \circ F$ are isomorphic functors.

Remark 2.5. Note that the binary relation $\sim$ is an equivalence relation on the class of actions. It is clearly reflexive; transitivity and symmetry follow straightforwardly from the fact that, for small categories C and D, if $F\colon C \longrightarrow D$ and $G\colon C \longrightarrow D$ are isomorphic functors, then FH and GH are also isomorphic, for every functor H from a small category into C.

Definition 2.6. Consider an action of a semigroup S on a set Q. Let $\mathbb{A}_Q$ be the action of $\mathbb{K}(S)$ on Q such that $\mathbb{A}_Q(e) = Qe$ for every object e of $\mathbb{K}(S)$, and such that $\mathbb{A}_Q((e,s,f))$ is the function $Qe \longrightarrow Qf$ mapping q to qs, for every $q \in Qe$. That is, $q \cdot (e,s,f) = q \cdot s$.

Lemma 2.7 ([20])**.** *If the action of the semigroup S on Q is faithful, then the action $\mathbb{A}_Q$ is faithful.*

3. Symbolic dynamics

3.1. **Shifts.** A good reference for the notions that we shall use here from symbolic dynamics is [52]. To make the paper reasonably self-contained and to introduce notation, we recall some basic definitions.

Let A be a finite alphabet, and consider the set $A^{\mathbb{Z}}$ of all bi-infinite sequences over A. The *shift* on $A^{\mathbb{Z}}$ is the homeomorphism $\sigma_A\colon A^{\mathbb{Z}} \longrightarrow A^{\mathbb{Z}}$ (or just σ) defined by $\sigma_A((x_i)_{i\in\mathbb{Z}}) = (x_{i+1})_{i\in\mathbb{Z}}$. We endow $A^{\mathbb{Z}}$ with the product topology with respect to the discrete topology on A. In particular, $A^{\mathbb{Z}}$ is a compact totally disconnected space (we include the Hausdorff property in the definition of compact).

We assume henceforth that all alphabets are finite. A *symbolic dynamical system* is a non-empty closed subset $\mathscr{X}$ of $A^{\mathbb{Z}}$, for some alphabet A, such that $\sigma(\mathscr{X}) = \mathscr{X}$. Symbolic dynamical systems are also known as *shift spaces*, *subshifts*, or, more plainly, just *shifts*. We shall generally prefer the latter option except when we are emphasizing that one shift is a subshift of another. The *orbit* of $x \in A^{\mathbb{Z}}$ is the set $\mathcal{O}(x) = \{\sigma^n(x) : n \in \mathbb{Z}\}$. Hence $\mathscr{X} \subseteq A^{\mathbb{Z}}$ is a shift if and only if $\mathscr{X}$ contains the orbits of its elements and it is topologically closed. When $\mathcal{O}(x)$ is finite then x is *periodic*, and $\mathcal{O}(x)$ is a *periodic* shift.

We can consider the category of shifts, whose objects are the shifts and where a morphism between two shifts $\mathscr{X} \subseteq A^{\mathbb{Z}}$ and $\mathscr{Y} \subseteq B^{\mathbb{Z}}$ is a continuous function $\Phi\colon \mathscr{X} \longrightarrow \mathscr{Y}$ such that $\Phi \circ \sigma_A = \sigma_B \circ \Phi$. In this category, an isomorphism is called a *conjugacy*. Isomorphic shifts are said to be *conjugate*.

The free semigroup and the free monoid over an alphabet A are denoted by A^+ and A^*, respectively. The length of an element $w \in A^*$ is denoted $|w|$. Recall that a subset of A^+ is in this context called a *language*.

A *block* of an element $(x_i)_{i\in\mathbb{Z}}$ of $A^{\mathbb{Z}}$ is a word of the form $x_i x_{i+1} \cdots x_{i+n-1} x_{i+n}$ (briefly denoted by $x_{[i,i+n]}$), where $i \in \mathbb{Z}$ and $n \geq 0$. If $\mathscr{X}$ is a subset of $A^{\mathbb{Z}}$ then we denote by $L(\mathscr{X})$ the language of blocks of elements of $\mathscr{X}$. If $\mathscr{X}$ is a subshift of $A^{\mathbb{Z}}$ and $x \in A^{\mathbb{Z}}$, then $x \in \mathscr{X}$ if and only if $L(\{x\}) \subseteq L(\mathscr{X})$ [52, Corollary 1.3.5].

A language $L \subseteq A^+$ is *factorial* if it is closed under taking factors, and it is *prolongable* if, for every element u of L there are elements $a, b \in A$ such that $aub \in L$. The correspondence $\mathscr{X} \mapsto L(\mathscr{X})$ is an order isomorphism between the poset of subshifts of $A^{\mathbb{Z}}$ and the poset of non-empty factorial, prolongable languages in A^+ [52, Proposition 1.3.4].

3.2. **The syntactic semigroup of a shift.** We shall make use of several well-known fundamental ideas and facts about the interplay between finite automata, formal languages and semigroups which, with some variations, can be found in several books, such as [24, 65].

If A is an alphabet, let $A_0^+ = A^+ \cup \{0\}$ be the free semigroup on A with zero. The multiplication of A^+ is extended to make 0 a zero element. Let $L \subseteq A^+$ be a language and let $u \in A_0^+$. The *context of* u *in* L, which we denote by $\delta_L(u)$, is the set $\{(x, y) \in A^* \colon xuy \in L\}$. Of course, $\delta_L(0) = \emptyset$. The relation $\equiv_L$ on A_0^+ defined by

$$u \equiv_L v \iff \delta_L(u) = \delta_L(v)$$

is a semigroup congruence, and the quotient semigroup $A_0^+/{\equiv_L}$ is the *syntactic semigroup (with zero) of* L, denoted $S(L)$. Note that the class of 0 is the zero element of $S(L)$. Since $u \equiv_L v$ if and only if $\delta_L(u)$ and $\delta_L(v)$ are equal, we may identify $u/{\equiv_L}$ with $\delta_L(u)$, and so δ_L can be viewed as the syntactic homomorphism $A_0^+ \longrightarrow S(L)$.

For all $u, v, w \in S(L)$, if $\delta_L(u) \subseteq \delta_L(v)$ then $\delta_L(wu) \subseteq \delta_L(wv)$ and $\delta_L(uw) \subseteq \delta_L(vw)$. This motivates the more refined notion of *syntactic ordered semigroup of a language* [66], which in this paper is necessary only for the discussion in Appendix A.

For a subshift $\mathscr{X}$ of $A^{\mathbb{Z}}$, we use the notations $\delta_{\mathscr{X}}$, $S(\mathscr{X})$, $LU(\mathscr{X})$ instead of $\delta_{L(\mathscr{X})}$, $S(L(\mathscr{X}))$ and $LU(S(\mathscr{X}))$ respectively. We say that $S(\mathscr{X})$ is the *syntactic semigroup of* $\mathscr{X}$. One has $u \in A_0^+ \setminus L(\mathscr{X})$ if and only if $\delta_{\mathscr{X}}(u) = \emptyset$. If $\mathscr{X} \subseteq A^{\mathbb{Z}}$ and $A \subseteq B$, then the syntactic semigroups of $\mathscr{X}$ viewed as a subset of $A^{\mathbb{Z}}$ and of $B^{\mathbb{Z}}$ coincide.

3.3. **Labeled graphs and sofic shifts.**

3.3.1. *Labeled graphs.* In this paper, *graph* will always mean a multi-edge directed graph. By a *labeled graph (over an alphabet A)* we mean a pair (G, λ) consisting of a graph G and a map $\lambda \colon E(G) \longrightarrow A$, where $E(G)$ is the set of edges of G, such that if e and f are distinct edges with the same origin and the same terminus, then $\lambda(e) \neq \lambda(f)$. The letter $\lambda(e)$ is the *label* of e. The labeled graph (G, λ) is *right resolving* if distinct edges with the same origin have distinct labels. It is *complete* if, for every vertex q and every letter $a \in A$, there is an edge starting in q with label a. Note that a complete labeled graph over A may be viewed as an action of A^+ over the set of vertices: the existence of an edge from q to r labeled a corresponds to the equality $q \cdot a = r$.

3.3.2. *Minimal automaton.* An *automaton* is just a labeled graph (G,λ) together with some sets I and F of vertices of G, respectively the set of *initial vertices* and the set of *final vertices*. A *deterministic* automaton is a right resolving automaton with a single initial vertex. The set of words labeling paths from vertices of I to vertices of F is the language *recognized* by the automaton. We view a labeled graph as an automaton in which all states are initial and final.

Consider a language $L \subseteq A^+$. Let u be an element of $A_0^* = A^* \cup \{0\}$, the free monoid with zero. The *right context of* u *in* L is the set

$$R_L(u) = \{w \in A^* \mid uw \in L\}.$$

Viewing L as a language over the alphabet $A \cup \{0\}$, we can consider its *minimal complete deterministic automaton*, which is the terminal object on the category of complete deterministic automatons, over the alphabet $A \cup \{0\}$, recognizing L. This automaton, which we denote by $\mathfrak{M}(L)$, can be realized as follows: the states are the right contexts of L, the initial state is $R_L(1)$, the final states are the right contexts $R_L(u)$ such that $u \in L$, and the action of an element v of A_0^+ on the set of states is given by $R_L(u) \cdot v = R_L(uv)$. A *sink* in a labeled graph is vertex z such that all edges starting in z are loops; the vertex $R_L(0) = \emptyset$ has the particularity of being the unique sink of $\mathfrak{M}(L)$. The language L is *recognizable* if it can be recognized by a finite automaton; L is recognizable if and only if $\mathfrak{M}(L)$ is finite, if and only if $S(L)$ is finite [24, 65].

Note that $u, v \in A_0^+$ have the same action over the states of $\mathfrak{M}(L)$ if and only if $\delta_L(u) = \delta_L(v)$. In particular, we may consider the action of $S(L)$ on the set of states defined by $R_L(u) \cdot \delta_L(v) = R_L(uv)$.

For a subshift $\mathscr{X}$ of $A^{\mathbb{Z}}$, we use the notation $\mathfrak{M}(\mathscr{X})$ instead of $\mathfrak{M}(L(\mathscr{X}))$.

3.3.3. *Sofic shifts.* A graph G is *essential* if the in-degree and the out-degree of each vertex is at least one. If the shift $\mathscr{X}$ and the essential labeled graph (G,λ) are such that $L(\mathscr{X})$ is recognized by (G,λ), then we say that $\mathscr{X}$ is the *shift presented by* (G,λ).

The shifts that can be presented by a finite labeled graph are called *sofic* [52, Chapter 3]. The sofic shifts are the shifts $\mathscr{X}$ such that $L(\mathscr{X})$ is a recognizable language. That is, $\mathscr{X}$ is sofic if and only if $S(\mathscr{X})$ is finite. The most studied class of sofic shifts is that of *finite type shifts* [52, Chapter 2]: a subshift $\mathscr{X}$ of $A^{\mathbb{Z}}$ is of finite type when $L(\mathscr{X}) = A^+ \setminus A^*WA^*$, for some finite subset W of A^+. An *edge shift* is a shift presented by a finite essential labeled graph (G,λ) such that the mapping λ is one-to-one. One of the characterizations of the shifts of finite type is that they are the shifts conjugate to edge shifts [52, Theorem 2.3.2].

A subshift $\mathscr{X}$ of $A^{\mathbb{Z}}$ is *irreducible* if for all $u, v \in L(\mathscr{X})$ there is $w \in A^*$ such that $uwv \in L(\mathscr{X})$. A sofic shift is irreducible if and only if it can be presented by a strongly connected labeled graph [52, Proposition 3.3.11].

3.4. **Synchronizing shifts.** Let $\mathscr{X}$ be a shift. A word u of $L(\mathscr{X})$ is *synchronizing*[1] if, whenever $vu \in L(\mathscr{X})$ and $uw \in L(\mathscr{X})$, we have $vuw \in L(\mathscr{X})$. An irreducible shift $\mathscr{X}$ is *synchronizing* if $L(\mathscr{X})$ contains a synchronizing word. Every irreducible sofic shift is synchronizing; this fact goes back to [26, Lemma 2], where the terminology "synchronizing" is absent; a generalization with that terminology appears in [9, Proposition 3.1].

The following lemma about synchronizing words can be useful.

Lemma 3.1. *Let $\mathscr{X} \subseteq A^{\mathbb{Z}}$ be a synchronizing shift.*

(1) The synchronizing words form an ideal of A^+.
(2) A synchronizing word u maps to an idempotent of $S(\mathscr{X})$ if and only if $u^2 \in L(\mathscr{X})$.
(3) If v is synchronizing, then $uv \in L(\mathscr{X})$ implies that $R_{\mathscr{X}}(uv) = R_{\mathscr{X}}(v)$.

Proof. Suppose that u is synchronizing and let $r \in A^+$. Then if $vru, ruw \in L(\mathscr{X})$, one has that $(vr)uw \in L(\mathscr{X})$ because u is synchronizing. Thus ru is synchronizing. Similarly, ur is synchronizing and so the synchronizing words form an ideal. This proves the first item.

Suppose that u maps to an idempotent of $S(\mathscr{X})$. Then trivially $u^2 \in L(\mathscr{X})$. For the converse, suppose that $u^2 \in L(\mathscr{X})$. If $v, w \in A^*$ with $vuw \in L(\mathscr{X})$, then because $vu, uu \in L(\mathscr{X})$, we have $vuu \in L(\mathscr{X})$. But then $vuu, uw \in L(\mathscr{X})$ implies that $vu^2w \in L(\mathscr{X})$. Conversely, if $vu^2w \in L(\mathscr{X})$, then $vu, uw \in L(\mathscr{X})$ and so $vuw \in L(\mathscr{X})$ because u is synchronizing.

The final statement follows because $vw \in L(\mathscr{X})$ if and only if $uvw \in L(\mathscr{X})$ by the definition of a synchronizing word. □

The main results of this paper concern the Karoubi envelope of $S(\mathscr{X})$, which has at least one object, namely 0. The following proposition establishes in particular that being synchronizing is a sufficient condition for the existence of other objects.

Proposition 3.2. *Let $\mathscr{X}$ be a synchronizing shift. If u is a synchronizing word of $L(\mathscr{X})$, then there is an idempotent $e \in S(\mathscr{X}) \setminus \{0\}$ such that $R_{\mathscr{X}}(1) \cdot e = R_{\mathscr{X}}(u)$.*

Proof. Suppose that u is a synchronizing word of $L(\mathscr{X})$. Since $\mathscr{X}$ is irreducible, there is a word v such that $uvu \in L(\mathscr{X})$. Then $vu, uvu \in L(\mathscr{X})$ implies that $vuvu \in L(\mathscr{X})$. By Lemma 3.1, vu is synchronizing and $e = \delta_{\mathscr{X}}(vu)$ is a non-zero idempotent. Lemma 3.1 also yields $R_{\mathscr{X}}(u) = R_{\mathscr{X}}(vu) = R_{\mathscr{X}}(1) \cdot e$. □

The condition given in Proposition 3.2 is not necessary for the existence of idempotents in $S(\mathscr{X}) \setminus \{0\}$, as testified by the class of irreducible shifts analyzed in Section 9. In the next lemma we state a necessary condition.

[1]In [52], a synchronizing word is called an *intrinsically synchronizing* word. There is some diversity of terminology in the literature (cf. [3, Remark 2.6]).

Lemma 3.3. *If $S(\mathscr{X})$ contains a non-zero idempotent, then $\mathscr{X}$ contains a periodic element.*

Proof. If $\delta_{\mathscr{X}}(u)$ is idempotent and $\delta_{\mathscr{X}}(u) \neq 0$, then $u^n \in L(\mathscr{X})$ for all n, which implies that $\cdots uuu.uuu \cdots$ is a periodic element of $\mathscr{X}$. $\square$

3.5. **Krieger and Fischer covers.** Denote the set of negative integers by $\mathbb{Z}^-$, and the set of non-negative integers by $\mathbb{N}$. Given an element $x = (x_i)_{i\in\mathbb{Z}^-}$ of $A^{\mathbb{Z}^-}$ and an element $y = (y_i)_{i\in\mathbb{N}}$ of $A^{\mathbb{N}}$, denote by $x.y$ the element $z = (z_i)_{i\in\mathbb{Z}}$ of $A^{\mathbb{Z}}$ for which $z_i = x_i$ if $i < 0$, and $z_i = y_i$ if $i \geq 0$.

If $x \in A^{\mathbb{Z}^-}$ and $u = a_1 \cdots a_n \in A^+$, with $a_i \in A$ when $1 \leq i \leq n$, then xu denotes the element of $A^{\mathbb{Z}^-}$ given by the left-infinite sequence $\cdots x_{-3}x_{-2}x_{-1}a_1 \cdots a_n$. Similarly, if $x \in A^{\mathbb{N}}$ then $ux \in A^{\mathbb{N}}$ is given by the right-infinite sequence $a_1 \cdots a_n x_0 x_1 x_2 \cdots$.

Let $\mathscr{X}$ be a subshift of $A^{\mathbb{Z}}$. We shall consider the sets

$$\mathscr{X}^- = \{x \in A^{\mathbb{Z}^-} \mid \exists y \in A^{\mathbb{N}} : x.y \in \mathscr{X}\}$$

and

$$\mathscr{X}^+ = \{x \in A^{\mathbb{N}} \mid \exists y \in A^{\mathbb{Z}^-} : y.x \in \mathscr{X}\}.$$

For each $x \in A^{\mathbb{Z}^-}$, let

$$C_{\mathscr{X}}(x) = \{y \in A^{\mathbb{N}} : x.y \in \mathscr{X}\}.$$

Note that $C_{\mathscr{X}}(x) \neq \emptyset$ if and only if $x \in \mathscr{X}^-$.

If $u \in A^+$, then $C_{\mathscr{X}}(x) = C_{\mathscr{X}}(z)$ implies $C_{\mathscr{X}}(xu) = C_{\mathscr{X}}(zu)$. This enables the following definition.

Definition 3.4 (Krieger cover)**.** Let $Q(\mathscr{X}) = \{C_{\mathscr{X}}(x) \mid x \in A^{\mathbb{Z}^-}\} \cup \{\emptyset\}$. Denote by $\mathfrak{K}^0(\mathscr{X})$ the right resolving complete labeled graph over $A \cup \{0\}$, with vertex set $Q(\mathscr{X})$, defined by the action of A_0^+ on $Q(\mathscr{X})$ given by

$$C_{\mathscr{X}}(x) \cdot u = C_{\mathscr{X}}(xu) \quad \text{if } u \in A^+, \quad C_{\mathscr{X}}(x) \cdot 0 = \emptyset,$$

having $\emptyset$ as unique sink. The labeled graph $\mathfrak{K}(\mathscr{X})$ over A obtained from $\mathfrak{K}^0(\mathscr{X})$ by elimination of the vertex $\emptyset$ is the *right Krieger cover of $\mathscr{X}$* (cf. [25, Definition 0.11]).

Krieger introduced in [45] this cover for sofic shifts only. If $\mathscr{X}$ is sofic, then the right Krieger cover of $\mathscr{X}$ embeds in the automaton $\mathfrak{M}(\mathscr{X})$ and it is computable [5, Section 4]. There are examples of synchronizing shifts whose Krieger graph is uncountable (cf., the example in the proof of [25, Corollary 1.3]). Hence, in the non-sofic case, the Krieger cover may not be a labeled subgraph of $\mathfrak{M}(\mathscr{X})$, which is always at most countable.

It is convenient for what follows to describe the syntactic congruence of $L(\mathscr{X})$ in terms of infinite words. Define, for $u \in A_0^+$,

$$\Delta_{\mathscr{X}}(u) = \{(x, y) \in A^{\mathbb{Z}^-} \times A^{\mathbb{N}} : xu.y \in \mathscr{X}\}$$

where we understand $\Delta_{\mathscr{X}}(0) = \emptyset$. The next proposition shows that $\Delta_{\mathscr{X}}(u)$ contains the same information as $\delta_{\mathscr{X}}(u)$.

Proposition 3.5. *Let $u, v \in A_0^+$. Then $\delta_{\mathscr{X}}(u) \subseteq \delta_{\mathscr{Y}}(v)$ if and only if $\Delta_{\mathscr{X}}(u) \subseteq \Delta_{\mathscr{Y}}(v)$.*

Proof. Suppose first that $\delta_{\mathscr{X}}(u) \subseteq \delta_{\mathscr{Y}}(v)$ and that $xu.y \in \mathscr{X}$ with $x \in A^{\mathbb{Z}^-}$ and $y \in A^{\mathbb{N}}$. Then $x_{[-n,-1]}uy_{[0,n]} \in L(\mathscr{X})$ for all $n \geq 1$ and so by assumption $x_{[-n,-1]}vy_{[0,n]} \in L(\mathscr{X})$ for all $n \geq 1$. As n was arbitrary, it follows that $xv.y \in \mathscr{X}$.

Conversely, suppose that $\Delta_{\mathscr{X}}(u) \subseteq \Delta_{\mathscr{Y}}(v)$ and that $rus \in L(\mathscr{X})$ with $r, s \in A^*$. Then $xru.sy \in \mathscr{X}$ for some $x \in A^{\mathbb{Z}^-}$ and $y \in A^{\mathbb{N}}$. Thus by hypothesis, we have $xrv.sy \in \mathscr{X}$ and hence $rvs \in L(\mathscr{X})$, as required. □

This proposition allows us to prove the following lemma, which will be put to use in proving that $S(\mathscr{X})$ acts on the Krieger cover.

Lemma 3.6. *Consider a subshift $\mathscr{X}$ of $A^{\mathbb{Z}}$, and let $u, v \in A^+$ and $x \in A^{\mathbb{Z}^-}$. If $\delta_{\mathscr{X}}(u) \subseteq \delta_{\mathscr{X}}(v)$ then $C_{\mathscr{X}}(xu) \subseteq C_{\mathscr{X}}(xv)$.*

Proof. By Proposition 3.5, we have that $\Delta_{\mathscr{X}}(u) \subseteq \Delta_{\mathscr{X}}(v)$. Suppose that $y \in C_{\mathscr{X}}(xu)$. Then $xu.y \in \mathscr{X}$ and hence $xv.y \in \mathscr{X}$ by definition of $\Delta_{\mathscr{X}}$. Thus $y \in C_{\mathscr{X}}(xv)$ and so $C_{\mathscr{X}}(xu) \subseteq C_{\mathscr{X}}(xv)$. □

By Lemma 3.6, the action of $S(\mathscr{X})$ on the set of vertices of $\mathfrak{K}^0(\mathscr{X})$ given by $C_{\mathscr{X}}(x) \cdot \delta_{\mathscr{X}}(u) = C_{\mathscr{X}}(xu)$ is well defined. According to the next lemma, this action is faithful; more precisely, it is faithful on the set of vertices of $\mathfrak{K}(\mathscr{X})$.

Lemma 3.7. *Let $u, v \in A^+$ be such that $C_{\mathscr{X}}(xu) \subseteq C_{\mathscr{X}}(xv)$ for all $x \in \mathscr{X}^-$. Then $\delta_{\mathscr{X}}(u) \subseteq \delta_{\mathscr{X}}(v)$.*

Proof. By Proposition 3.5, it suffices to prove $\Delta_{\mathscr{X}}(u) \subseteq \Delta_{\mathscr{X}}(v)$. Suppose that $xu.y \in \mathscr{X}$. Then $xv.y \in \mathscr{X}$ because $C_{\mathscr{X}}(xu) \subseteq C_{\mathscr{X}}(xv)$. This completes the proof. □

We shall also consider sets of the following form, for a block $u \in L(\mathscr{X})$:

$$C_{\mathscr{X}}(u) = \{y \in A^{\mathbb{N}} : uy \in \mathscr{X}^+\}.$$

Note that if $\mathscr{X}$ is synchronizing, and u is a synchronizing block of $L(\mathscr{X})$, then $C_{\mathscr{X}}(u) = C_{\mathscr{X}}(xu)$ for all $x \in A^*$ with $xu \in L(\mathscr{X})$.

In a graph G, a subgraph H is *terminal* if every edge of G starting in a vertex of H belongs to H. A *strongly connected component* of G is a maximal strongly connected subgraph of G.

As a reference for the next definition, which we are ready to state, we give Definition 0.12 in [25], and the lines following it in [25], for a justification that it is well done.

Definition 3.8 (Fischer cover)**.** Let $\mathscr{X}$ be a synchronizing shift. If u and v are synchronizing words of $L(\mathscr{X})$, then $C_{\mathscr{X}}(u)$ and $C_{\mathscr{X}}(v)$ are vertices in the same strongly connected component of $\mathfrak{K}(\mathscr{X})$. This strongly connected component is terminal and it is a presentation of $\mathscr{X}$. It is called the *right Fischer cover of* $\mathscr{X}$, and we denote it by $\mathfrak{F}(\mathscr{X})$.

We denote by $\mathfrak{F}^0(\mathscr{X})$ the terminal complete labeled subgraph of $\mathfrak{K}^0(\mathscr{X})$ obtained from $\mathfrak{F}(\mathscr{X})$ by adjoining the sink state $\{\emptyset\}$. We denote by $Q_{\mathfrak{F}}(\mathscr{X})$ the vertex set of $\mathfrak{F}^0(\mathscr{X})$.

Given a labeled graph (G, λ) over an alphabet A, one says that a word $u \in A^+$ is a *synchronizing word for* (G, λ) if u is recognized by (G, λ) and all paths in (G, λ) labeled by u end in the same vertex[2]. A labeled graph is *reduced* if it has the following property: if p and q are vertices such that the set of words labeling paths starting in p is equal to the set of words labeling paths starting in q, then $p = q$. A finite labeled graph which is right resolving and reduced has a synchronizing word [52, Proposition 3.3.16].

The following result is well known, cf. [25, Theorem 2.16] and [52, Corollary 3.3.19].

Theorem 3.9. *If $\mathscr{X}$ is a synchronizing shift, then $\mathfrak{F}(\mathscr{X})$ is the unique labeled graph (up to isomorphism of labeled graphs) which is strongly connected, reduced, right resolving and for which there is a synchronizing word.*

The next result was proved in [8] for irreducible sofic shifts. The generalization for synchronizing shifts offers no additional difficulty, but we include here its proof for the sake of completeness. We denote by $\mathfrak{M}^*(\mathscr{X})$ the labeled graph obtained from $\mathfrak{M}(\mathscr{X})$ by eliminating the sink vertex $\emptyset$.

Proposition 3.10. *Let $\mathscr{X}$ be a synchronizing shift. Then the labeled graph $\mathfrak{M}^*(\mathscr{X})$ has a unique terminal strongly connected component, which is isomorphic with $\mathfrak{F}(\mathscr{X})$. Its vertices are the right-contexts of synchronizing words.*

Proof. Let $u, v \in L(\mathscr{X})$ with v synchronizing. Since $\mathscr{X}$ is irreducible, there is $w \in L(\mathscr{X})$ such that $uwv \in L(\mathscr{X})$. Then $R_{\mathscr{X}}(u)wv = R_{\mathscr{X}}(uwv) = R_{\mathscr{X}}(v)$ by Lemma 3.1. Thus $\mathfrak{M}^*(\mathscr{X})$ has a unique terminal strongly connected component G and $R_{\mathscr{X}}(v) \in G$ for every synchronizing word v. Moreover, since the synchronizing words form an ideal by Lemma 3.1, all vertices of G are of this form.

To show that G is isomorphic with $\mathfrak{F}(\mathscr{X})$, we verify the conditions in Theorem 3.9. That G presents $\mathscr{X}$ follows straightforwardly from $\mathscr{X}$ being irreducible. Every labeled subgraph of $\mathfrak{M}^*(\mathscr{X})$ is right resolving. Clearly, $\mathfrak{M}^*(\mathscr{X})$ is reduced. Since G is a terminal subgraph of $\mathfrak{M}^*(\mathscr{X})$, it inherits the property of being reduced. Finally, if v is a synchronizing word of $L(\mathscr{X})$, then all paths in G labeled by v end in $R_{\mathscr{X}}(v)$, thus v is a synchronizing word for $\mathfrak{F}(\mathscr{X})$. Hence G and $\mathfrak{F}(\mathscr{X})$ are isomorphic by Theorem 3.9. □

We have so far defined the *right* Krieger and Fischer covers. The *left* Krieger and Fischer covers are defined analogously, changing directions when needed. Equivalently, we can use the approach used in [60, pages 568–569] (see also [52, page 39]) to define the left Krieger and Fischer covers by means

[2]This terminology is from [52]. In [25], an important reference concerning synchronizing shifts, a synchronizing word for (G, λ) is referred as a *magic word of* (G, λ).

of the transpose operator. The *transpose* or *time reversal* of a shift $\mathscr{X}$ is the shift

$$\mathscr{X}^T = \{(x_i)_{i\in\mathbb{Z}} \mid (x_{-i})_{i\in\mathbb{Z}} \in \mathscr{X}\}.$$

The transpose of a labeled graph G is the labeled graph G^T obtained from G by reversing the directions of the edges, maintaining the labels. Then the left Krieger and Fischer covers of a shift $\mathscr{X}$ (the latter if $\mathscr{X}$ is synchronizing) are the labeled graphs $\mathfrak{K}(\mathscr{X}^T)^T$ and $\mathfrak{F}(\mathscr{X}^T)^T$, respectively.

3.6. **Flow equivalence.** Fix an alphabet A and a letter α of A. Let $\diamond$ be a letter not in A. Denote by B the alphabet $A\cup\{\diamond\}$. The *symbol expansion of* A^* *associated to* α is the unique monoid homomorphism $\mathcal{E}\colon A^* \longrightarrow B^*$ such that $\mathcal{E}(\alpha) = \alpha\diamond$ and $\mathcal{E}(a) = a$ for all $a \in A \setminus \{\alpha\}$. Note that $\mathcal{E}$ is injective.

The *symbol expansion of a subshift* $\mathscr{X}$ *of* $A^{\mathbb{Z}}$ *relatively to* α is the least subshift $\mathscr{X}'$ of $B^{\mathbb{Z}}$ such that $L(\mathscr{X}')$ contains $\mathcal{E}(L(\mathscr{X}))$. A *symbol expansion of* $\mathscr{X}$ is a symbol expansion of $\mathscr{X}$ relatively to some letter.

The mapping $\mathcal{E}$ admits the following natural extension of its domain and range. If $x \in A^{\mathbb{Z}^-}$ and $y \in A^{\mathbb{N}}$, then $\mathcal{E}(x)$ and $\mathcal{E}(y)$ are respectively the elements of $B^{\mathbb{Z}^-}$ and $B^{\mathbb{N}}$ given by

$$\mathcal{E}(x) = \cdots \mathcal{E}(x_{-3})\,\mathcal{E}(x_{-2})\,\mathcal{E}(x_{-1}), \qquad \mathcal{E}(y) = \mathcal{E}(y_0)\,\mathcal{E}(y_1)\,\mathcal{E}(y_2)\cdots .$$

Moreover, $\mathcal{E}(x.y)$ denotes $\mathcal{E}(x).\mathcal{E}(y) \in B^{\mathbb{Z}}$. Note that $\mathscr{X}'$ is the least subshift of $B^{\mathbb{Z}}$ containing $\mathcal{E}(\mathscr{X})$.

The *flow equivalence* is the least equivalence relation between shifts containing the conjugacy and symbol expansion relations. The classes of finite type shifts, of sofic shifts, and of irreducible shifts are all easily seen to be closed under flow equivalence. See [52, Section 13.6] for motivation for studying flow equivalence. Here, we just remark that the original definition of flow equivalence (that two shifts are flow equivalent if their suspension flows are topologically equivalent) was proved in [61] to be equivalent to the one we use, explicitly for finite type shifts, but as pointed out in [56, Lemma 2.1], implicitly for all shifts. See also [62, page 87].

Note that $\mathscr{X}$ and $\mathscr{Y}$ are flow equivalent shifts if and only if $\mathscr{X}^T$ and $\mathscr{Y}^T$ are flow equivalent. Hence, for the purposes of our paper, it is not necessary to translate results on right Krieger and Fischer covers to dual results on left Krieger and Fischer covers.

4. Statement of the main results

In this section we state the main results of this paper, deferring proofs to the final sections.

Definition 4.1 (Karoubi envelope of a shift)**.** If $\mathscr{X}$ is a shift, then the *Karoubi envelope* $\mathbb{K}(\mathscr{X})$ *of* $\mathscr{X}$ is by definition the Karoubi envelope $\mathbb{K}(S(\mathscr{X}))$ of its syntactic semigroup.

Note that the category $\mathbb{K}(\mathscr{X})$ is a category with zero, since $S(\mathscr{X})$ is a semigroup with zero. Our principal result is that the natural equivalence class of $\mathbb{K}(\mathscr{X})$ is a flow equivalence invariant of $\mathscr{X}$.

Theorem 4.2. *If $\mathscr{X}$ and $\mathscr{Y}$ are flow equivalent shifts, then the categories $\mathbb{K}(\mathscr{X})$ and $\mathbb{K}(\mathscr{Y})$ are equivalent, i.e., $S(\mathscr{X})$ and $S(\mathscr{Y})$ are Morita equivalent up to local units.*

Theorem 4.2 is a direct consequence of a more detailed result which is stated afterwards in Theorem 4.4. We remark that if $\mathscr{X}$ and $\mathscr{Y}$ are sofic shifts given by presentations, then one can effectively determine whether $\mathbb{K}(\mathscr{X})$ is equivalent to $\mathbb{K}(\mathscr{Y})$. This is because these categories are finite and effectively constructible and so one can in principle check all functors between them and see if there is one which is an equivalence.

The reader is referred back to Definition 2.6. For a shift $\mathscr{X}$, denote by $\mathbb{A}_{\mathscr{X}}$ the action $\mathbb{A}_{Q(\mathscr{X})}$ arising from the action of $LU(\mathscr{X})$ on $Q(\mathscr{X})$.

Remark 4.3. According to Lemma 3.7, the semigroup $S(\mathscr{X})$ acts faithfully on $Q(\mathscr{X})$. In particular, the subsemigroup $LU(\mathscr{X})$ acts faithfully on $Q(\mathscr{X})$, and so, by Lemma 2.7, the action $\mathbb{A}_{\mathscr{X}}$ is faithful. The faithfulness of $\mathbb{A}_{\mathscr{X}}$ is used in the proof of the following theorem.

Theorem 4.4. *If $\mathscr{X}$ and $\mathscr{Y}$ are flow equivalent shifts, then the actions $\mathbb{A}_{\mathscr{X}}$ and $\mathbb{A}_{\mathscr{Y}}$ are equivalent.*

Note that Theorem 4.4 immediately implies Theorem 4.2, by the definition of equivalence of actions. The proof of Theorem 4.4 is carried out in Section 12, after some preliminaries in Section 11.

Recall from Subsection 3.5 the definition of right Fischer cover and the relationship between $Q(\mathscr{X})$ and $Q_{\mathfrak{F}}(\mathscr{X})$, when $\mathscr{X}$ is a synchronizing shift. Recall in particular that the elements of the set $Q_{\mathfrak{F}}(\mathscr{X})\setminus\{\emptyset\}$ are the vertices in the unique strongly connected terminal component of the Krieger cover of $\mathscr{X}$. This motivates the following remark.

Remark 4.5. If $q \in Q_{\mathfrak{F}}(\mathscr{X})$, then $qs \in Q_{\mathfrak{F}}(\mathscr{X})$, for every $s \in S(\mathscr{X})$.

Therefore, the action of $LU(\mathscr{X})$ on $Q(\mathscr{X})$ restricts in a natural way to an action of $LU(\mathscr{X})$ on $Q_{\mathfrak{F}}(\mathscr{X})$, denoted $\mathbb{A}^{\mathfrak{F}}_{\mathscr{X}}$.

Theorem 4.6. *If $\mathscr{X}$ and $\mathscr{Y}$ are flow equivalent synchronizing shifts, then the actions $\mathbb{A}^{\mathfrak{F}}_{\mathscr{X}}$ and $\mathbb{A}^{\mathfrak{F}}_{\mathscr{Y}}$ are equivalent.*

The proof of Theorem 4.6 is deferred to Section 13.

The *minimal shifts* — shifts not strictly containing another shift — are of great importance, and in many aspects quite apart from sofic shifts (cf. [52, Section 13.7]). A minimal shift is sofic if and only if it is periodic. It follows from Lemma 3.3 that if $\mathscr{X}$ is a minimal non-periodic shift, then 0 is the unique idempotent of $S(\mathscr{X})$. Hence, in that case our main results have no applications. On the other hand, they do have meaningful consequences for sofic, synchronizing, and other classes of shifts. In the next few sections we examine some of these consequences.

5. The proper comunication graph

For a graph G, let $PC(G)$ be the set of non-trivial strongly connected components of G. Here we consider a strongly connected graph to be trivial if it consists of one vertex and no edges (a single vertex with some loop edges is deemed non-trivial). Consider in $PC(G)$ the partial order given by $C_1 \leq C_2$ if and only if there is in G a path from an element of C_1 to an element of C_2. Following the terminology of [3], the *proper communication graph of* G is the acyclic directed graph with vertex set $PC(G)$ and edge set given by the irreflexive relation $<$. It is proved in [3] that the proper communication graph of the right (left) Krieger cover of a sofic shift is a flow equivalence invariant. We shall see in this section that this result can naturally be seen as a consequence of Theorem 4.4, and in the process of doing this, we generalize it to arbitrary shifts.

Let $q \in Q(\mathscr{X})$. We say that $s \in S(\mathscr{X})$ *stabilizes* q if $q \cdot s = q$. Denote by E_q the set of idempotents that stabilize q. Note that, for every $q \in Q(\mathscr{X})$, we have $e \in E_q$ if and only if $q \in Q(\mathscr{X})e$. Let $I(\mathscr{X})$ be the set of vertices $q \in Q(\mathscr{X})$ such that E_q is non-empty, i.e., $I(\mathscr{X}) = Q(\mathscr{X}) \cdot E(S(\mathscr{X}))$. Endow $I(\mathscr{X})$ with the preorder $\preceq$ defined by $q \preceq r$ if and only if there is a path from q to r in $\mathfrak{K}^0(\mathscr{X})$. That is, $q \preceq r$ if and only if $r = \emptyset$ or there is a path from q to r in $\mathfrak{K}(\mathscr{X})$. Equivalently, $q \preceq r$ if and only if $r \in qLU(\mathscr{X})$. Denote by $\sim$ the equivalence relation defined by $p \sim q$ if and only if $p \preceq q$ and $q \preceq p$. The quotient poset $(I(\mathscr{X})/{\sim}, \leq)$ will be denoted by $P(\mathscr{X})$. We can identify $P(\mathscr{X})$, when convenient, with the set $\{qLU(\mathscr{X}) : q \in I(\mathscr{X})\}$ ordered by inclusion. Note that $\emptyset \cdot LU(\mathscr{X}) = \emptyset$.

Proposition 5.1. *For the shifts $\mathscr{X}$ and $\mathscr{Y}$, suppose we have a natural equivalence $F\colon \mathbb{K}(\mathscr{X}) \longrightarrow \mathbb{K}(\mathscr{Y})$ for which there is a natural isomorphism $\eta\colon \mathbb{A}_{\mathscr{X}} \Rightarrow \mathbb{A}_{\mathscr{Y}} \circ F$. The following conditions hold:*

1. *For every $q \in Q(\mathscr{X})$, if $e \in E_q$ then $F(e) \in E_{\eta_e(q)}$. Hence, if $q \in I(\mathscr{X})$ and $e \in E_q$, then $\eta_e(q) \in I(\mathscr{Y})$.*
2. *Let $q, r \in I(\mathscr{X})$. For every $e \in E_q$ and $f \in E_r$, one has $q \preceq r$ if and only if $\eta_e(q) \preceq \eta_f(r)$.*
3. *The mapping $\psi\colon P(\mathscr{X}) \longrightarrow P(\mathscr{Y})$, defined by $\psi(qLU(\mathscr{X})) = \eta_e(q)LU(\mathscr{Y})$, where $e \in E_q$, is a well-defined isomorphism of posets.*

Proof. First we prove (1). Let $e \in E_q$. Then $\eta_e(q) \in Q(\mathscr{Y})F(e)$ and so $F(e) \in E_{\eta_e(q)}$.

To prove (2), suppose that $q \preceq r$. Then there is morphism (e, s, f) of $\mathbb{K}(\mathscr{X})$ such that $q \cdot (e, s, f) = r$. This implies $\eta_e(q) \cdot F(e, s, f) = \eta_f(r)$, thus $\eta_e(q) \preceq \eta_f(r)$. Conversely, suppose that $\eta_e(q) \preceq \eta_f(r)$. Recall from (1) that $F(e) \in E_{\eta_e(q)}$ and $F(f) \in E_{\eta_f(r)}$. Therefore, there is a morphism $(F(e), t, F(f))$ of $\mathbb{K}(\mathscr{Y})$ such that $\eta_e(q) \cdot (F(e), t, F(f)) = \eta_f(r)$. Since F is full, there is a morphism (e, s, f) of $\mathbb{K}(\mathscr{X})$ such that $\eta_e(q) \cdot F(e, s, f) = \eta_f(r)$. That is, we have $\eta_f(q \cdot (e, s, f)) = \eta_f(r)$. Since η_f is injective (as η is a natural isomorphism), this shows that $q \preceq r$, concluding the proof of (2).

To prove (3), we only need to show that ψ is onto, since the result is then an immediate consequence of (2). Let $p \in I(\mathscr{Y})$. Take $f \in E_p$. Since F is essentially surjective, there is an idempotent $e \in S(\mathscr{X})$ such that f and $F(e)$ are isomorphic. If $(f, s, F(e))$ is an isomorphism and $r = p\cdot(f, s, F(e))$, then $p \sim r$. Note that $r \in Q(\mathscr{Y})F(e)$. Since η_e is bijective, we may consider the vertex $\eta_e^{-1}(r) \in Q(\mathscr{X})e$. Then, clearly $\psi(\eta_e^{-1}(r)LU(\mathscr{X})) = pLU(\mathscr{X})$, thus showing that ψ is onto. □

Corollary 5.2. *The poset $P(\mathscr{X})$ is a flow equivalence invariant.*

Proof. Let $\mathscr{X}$ and $\mathscr{Y}$ be flow equivalent shifts. By Theorem 4.4, the actions $\mathbb{A}_{\mathscr{X}}$ and $\mathbb{A}_{\mathscr{Y}}$ are equivalent. Therefore, $P(\mathscr{X})$ and $P(\mathscr{Y})$ are isomorphic, by Proposition 5.1. □

Remark 5.3. If $\mathscr{X}$ is a synchronizing shift then, by Proposition 3.10, the elements of $Q_{\mathfrak{F}}(\mathscr{X})$ are the sink vertex $\emptyset$ and the vertices of the form $R_{\mathscr{X}}(u)$, with u a synchronizing word. Therefore, by Proposition 3.2, every element of $Q_{\mathfrak{F}}(\mathscr{X})$ belongs to $I(\mathscr{X})$. Moreover, clearly $Q_{\mathfrak{F}}(\mathscr{X})$ is the disjoint union of two $\sim$-classes: the $\sim$-class $Q_{\mathfrak{F}}(\mathscr{X}) \setminus \{\emptyset\}$ of the vertices of $\mathfrak{F}(\mathscr{X})$, and the $\sim$-class $\{\emptyset\}$. The $\sim$-class $\{\emptyset\}$ is the minimum element of the poset $P(\mathscr{X})$, and $Q_{\mathfrak{F}}(\mathscr{X}) \setminus \{\emptyset\}$ is the minimum element of $P(\mathscr{X}) \setminus \{\{\emptyset\}\}$. Therefore, it follows from Proposition 5.1 that $q \in Q_{\mathfrak{F}}(\mathscr{X})$ if and only if $\eta_e(q) \in Q_{\mathfrak{F}}(\mathscr{Y})$, for every $e \in E_q$.

The following simple fact will be useful to specialize to sofic shifts.

Lemma 5.4. *Suppose that $\mathscr{X}$ is a sofic shift and q belongs to a non-trivial strongly connected component of $\mathfrak{K}^0(\mathscr{X})$. Then there is an idempotent $e \in S(\mathscr{X})$ such that $q = q \cdot e$, i.e., $q \in I(\mathscr{X})$.*

Proof. Since the strongly connected component to which q belongs is non-trivial, there is a word u labeling a non-empty loop rooted at q. Then $q = q \cdot \delta_{\mathscr{X}}(u^n)$ for all $n \geq 1$. On the other hand, since $S(\mathscr{X})$ is finite, there is $m \geq 1$ such that $\delta_{\mathscr{X}}(u)^m$ is idempotent. □

We can now add the following corollary to Proposition 5.1.

Corollary 5.5. *The proper communication graph of the right (left) Krieger cover of a sofic shift is a flow equivalence invariant.*

Proof. Clearly, for every shift $\mathscr{X}$, a $\sim$-class of $I(\mathscr{X})$ is contained in a non-trivial strongly connected component of $\mathfrak{K}^0(\mathscr{X})$. Conversely, by Lemma 5.4, if $\mathscr{X}$ is sofic then every non-trivial strongly connected component of $\mathfrak{K}^0(\mathscr{X})$ is contained in a $\sim$-class. Therefore, the posets $P(\mathscr{X})$ and $PC(\mathfrak{K}^0(\mathscr{X}))$ are equal, and so $PC(\mathfrak{K}^0(\mathscr{X}))$ is a flow invariant by Corollary 5.2. Since $PC(\mathfrak{K}(\mathscr{X})) = PC(\mathfrak{K}^0(\mathscr{X})) \setminus \{\{\emptyset\}\}$, this concludes the proof. □

In [37] one finds a complete characterization of the acyclic graphs that can be the proper communication graph of the left Krieger cover of an *almost finite type shift*, a special type of irreducible sofic shift which will receive some attention in Section 7.

6. The labeled preordered set of the $\mathcal{D}$-classes of $S(\mathscr{X})$

In this section we use Theorem 4.4 and a result from [20] to obtain a flow equivalence invariant (Theorem 6.3) which, as we shall observe, improves some related results from [7, 17]. We close the section with some examples.

6.1. **Abstract semigroup setting.** Given a semigroup S, let $\mathfrak{D}(S)$ be the set of $\mathcal{D}$-classes of S. Endow $\mathfrak{D}(S)$ with the preorder $\preceq$ such that, if D_1 and D_2 belong to $\mathfrak{D}(S)$, then $D_1 \preceq D_2$ if and only if there are $d_1 \in D_1$ and $d_2 \in D_2$ such that $d_1 \leq_{\mathcal{J}} d_2$. Note that if $\mathcal{D} = \mathcal{J}$ (for example, if S is finite), then $\preceq$ is a partial order.

If we assign to each element x of a preordered set P a label $\lambda_P(x)$ from some set, we obtain a new structure, called *labeled preordered set*. A morphism in the category of labeled preordered sets is a morphism $\varphi\colon P \longrightarrow Q$ of preordered sets such $\lambda_Q \circ \varphi = \lambda_P$.

For a semigroup S, assign to each element D of $\mathfrak{D}(S)$ the label $\lambda(D) = (\varepsilon, H)$ where $\varepsilon \in \{0,1\}$, with $\varepsilon = 1$ if and only if D is regular, and H is the Schützenberger group of D. We denote the labeled preordered set thus obtained by $\mathfrak{D}_\ell(S)$. By Remark 2.2, $\mathfrak{D}_\ell\big(LU(S)\big)$ is obtained from $\mathfrak{D}_\ell(S)$ by removing the $\mathcal{D}$-classes of S not contained in $LU(S)$.

Theorem 6.1 ([20])**.** *Let S and T be semigroups with local units. If $\mathbb{K}(S)$ and $\mathbb{K}(T)$ are equivalent, then $\mathfrak{D}_\ell(S)$ and $\mathfrak{D}_\ell(T)$ are isomorphic labeled preordered sets.*

Suppose that the semigroup S acts on a set Q. Then each element of S can be viewed as a transformation on Q. Recall that the *rank* of a transformation is the cardinal of its image, and that $\mathcal{J}$-equivalent elements of S have the same rank, as transformations of Q. We modify the labeled preordered set $\mathfrak{D}_\ell\big(S\big)$ as follows: for each $\mathcal{D}$-class D of S, instead of the label (ε, H), consider the label (ε, H, r), where r is the rank in Q of an element of D, viewed as an element of the transformation semigroup of Q defined by the action of S on Q. Denote by $\mathfrak{D}_Q\big(S\big)$ the resulting labeled poset.

Theorem 6.2 ([20])**.** *Let S and T be semigroups with local units. Suppose that S acts on the set Q, and T acts on the set R. If $\mathbb{A}_Q$ and $\mathbb{A}_R$ are equivalent actions, then $\mathfrak{D}_Q(S)$ and $\mathfrak{D}_R(T)$ are isomorphic labeled preordered sets.*

6.2. **Application to shifts.** Consider a shift $\mathscr{X}$. Recall that $S(\mathscr{X})$ acts faithfully as a transformation semigroup on the set $Q(\mathscr{X})$. We mention that if $\mathscr{X}$ is sofic and irreducible, then the restriction of this action to $Q_{\mathfrak{F}}(\mathscr{X})$ is also faithful [5, Proposition 4.8].

The sink state $\emptyset$ is not a vertex of the Krieger cover of $\mathscr{X}$. That is why, following analogous conventions found in [7, 17], we consider the rank of the partial action of $S(\mathscr{X})$ on $Q(\mathscr{X}) \setminus \{\emptyset\}$. Hence, in the labeled preordered set $\mathfrak{D}_{Q(\mathscr{X})}(LU(\mathscr{X}))$, we replace the label (ε, H, r) of each $\mathcal{D}$-class D by the

label $(\varepsilon, H, r-1)$. We denote by $\mathfrak{KD}(\mathscr{X})$ the resulting labeled preordered set.

Similarly, if $\mathscr{X}$ is synchronizing, then we replace in $\mathfrak{D}_{Q_{\mathfrak{F}}(\mathscr{X})}(LU(\mathscr{X}))$ the label (ε, H, r) of each $\mathcal{D}$-class D by the label $(\varepsilon, H, r-1)$, and denote by $\mathfrak{FD}(\mathscr{X})$ the resulting labeled preordered set.

Theorem 6.3. *For every shift $\mathscr{X}$, the labeled preordered set $\mathfrak{KD}(\mathscr{X})$ is a flow equivalence invariant. If $\mathscr{X}$ is synchronizing then $\mathfrak{FD}(\mathscr{X})$ is also a flow equivalence invariant.*

Proof. This is an immediate consequence of Theorems 4.4 and 6.2. □

Theorem 6.3 states in particular that $\mathfrak{D}_\ell\big(LU(\mathscr{X})\big)$ is a flow equivalence invariant, which could also be deduced by invoking Theorems 6.1 and 4.2. In the doctoral thesis [19] one finds a quite long and technical proof of the conjugacy invariance of $\mathfrak{D}_\ell\big(LU(\mathscr{X})\big)$. The invariance under conjugacy of $\mathfrak{KD}(\mathscr{X})$ when $\mathscr{X}$ is sofic (and of $\mathfrak{FD}(\mathscr{X})$ if $\mathscr{X}$ is moreover irreducible) was first proved in [17]. Forgetting the non-regular $\mathcal{D}$-classes, we get from $\mathfrak{KD}(\mathscr{X})$ and $\mathfrak{FD}(\mathscr{X})$ weaker conjugacy invariants of sofic shifts, first obtained in [7].

6.3. **Examples.**

Example 6.4. Let $\mathscr{X}$ and $\mathscr{Y}$ be the irreducible sofic shifts on the 14-letter alphabet $\{a_1, \ldots, a_{14}\}$ whose right Fischer covers are respectively represented in Figure 1. The dashed edges are those whose label appears only in one edge. Note that the right Fischer covers have the same underlying unlabeled

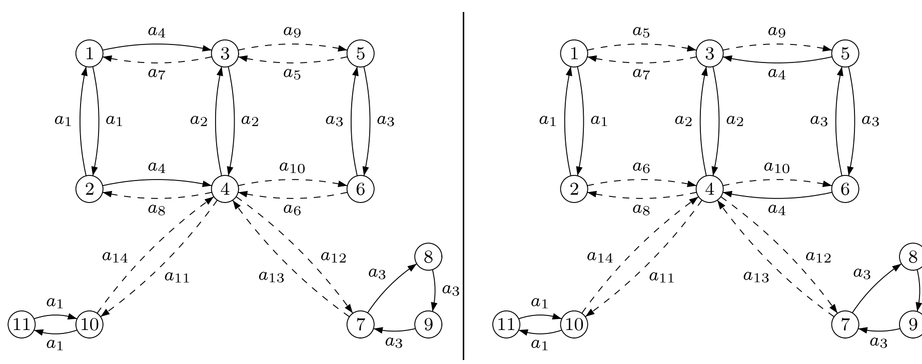

Figure 1. The irreducible sofic shifts $\mathscr{X}$ and $\mathscr{Y}$ in Example 6.4.

graph. The same phenomena occurs with the right Krieger covers of $\mathscr{X}$ and $\mathscr{Y}$, which are obtained from the labeled graphs in Figure 1 by glueing to them the corresponding graphs in Figure 2 (the additional vertices 12, 13 and 14 are represented by a square). In particular, they have the same proper communication graph.

The labeled ordered sets $\mathfrak{KD}(\mathscr{X})$ and $\mathfrak{KD}(\mathscr{Y})$ are represented in Figure 3 by their Hasse diagrams, where within each node of the diagram we find

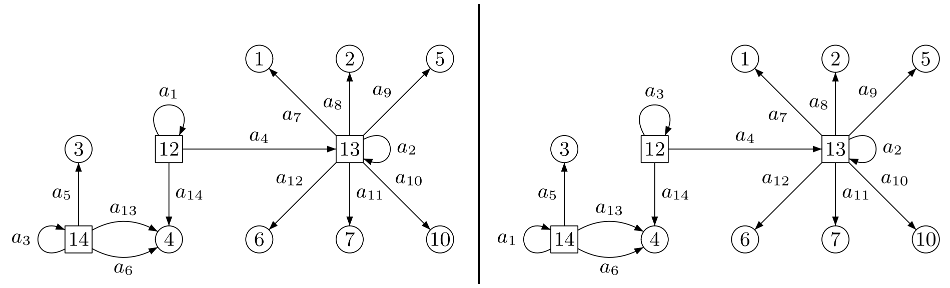

FIGURE 2. Edges added to obtain the right Krieger covers.

information identifying the $\mathcal{D}$-class and its label. For example, in the first diagram, the notation $a_2|(1, C_2, 3)$ means the node represents the $\mathcal{D}$-class of $\delta_{\mathscr{X}}(a_2)$ and that its label is the triple $(1, C_2, 3)$, where, as usual, C_n denotes the cyclic group of order n. The computations were carried out using GAP [30], more specifically the GAP packages *sgpviz* [22] and *automata* [23].

The labeled ordered sets $\mathfrak{KD}(\mathscr{X})$ and $\mathfrak{KD}(\mathscr{Y})$ are not isomorphic, hence $\mathscr{X}$ and $\mathscr{Y}$ are not flow equivalent.

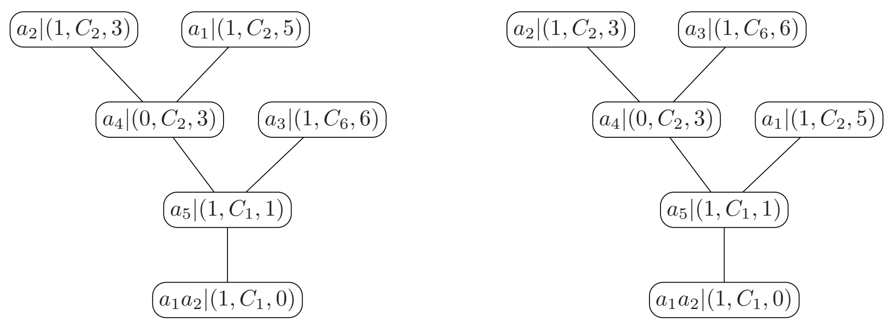

FIGURE 3. $\mathfrak{KD}(\mathscr{X})$ and $\mathfrak{KD}(\mathscr{Y})$.

Remark 6.5. Rune Johansen announced in his doctoral thesis [36] a result of Boyle, Carlsen and Eilers [14] stating that if two sofic shifts are flow equivalent then the edge shifts defined by the underlying unlabeled graphs of their right (left) Krieger covers are also flow equivalent, in a canonical way (cf. [36, Theorem 2.11 and Proposition 2.12]). In the case of irreducible sofic shifts, the same happens with the edge shifts defined by the right (left) Fischer covers. The Bowen-Franks groups of a sofic shift, as introduced in [56, 57], is encoded in the unlabeled graph underlying the Krieger cover. Information about the flow equivalence class is lost by taking the edge shifts only. That is what happens in Example 6.4, where the left (and also right) Krieger covers of $\mathscr{X}$ and $\mathscr{Y}$ have the same underlying unlabeled graph and the same happens with the Fischer covers.

Remark 6.6. The shifts from Example 6.4 appeared in the doctoral thesis of the first author [19, Example 6.25], where flow equivalence was not considered, as an example of a pair of sofic shifts in which the fact that they are not conjugate is revealed by the labeled preordered sets considered in Theorem 6.3. Other relevant conjugacy invariants, which are not flow equivalence invariants (like the *zeta function* [52, Section 6]) fail to separate them.

Example 6.7. Let $\mathscr{X}$ and $\mathscr{Y}$ be the irreducible sofic shifts respectively presented in Figure 4. The labeled graphs are clearly right resolving, strongly

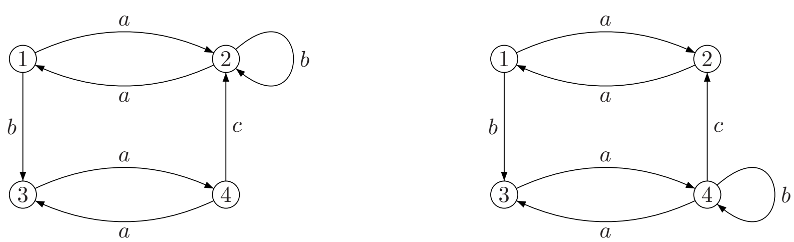

FIGURE 4. Two irreducible shifts $\mathscr{X}$ and $\mathscr{Y}$.

connected and reduced, whence they are right Fischer covers. Note that a^2 fixes all vertices, therefore $S(\mathscr{X})$ and $S(\mathscr{Y})$ are monoids.

The *eggbox* diagrams of $S(\mathscr{X})$ and $S(\mathscr{Y})$ are respectively represented in Figure 5, obtained with GAP package *sgpviz* [22]. The outer rectangles rep-

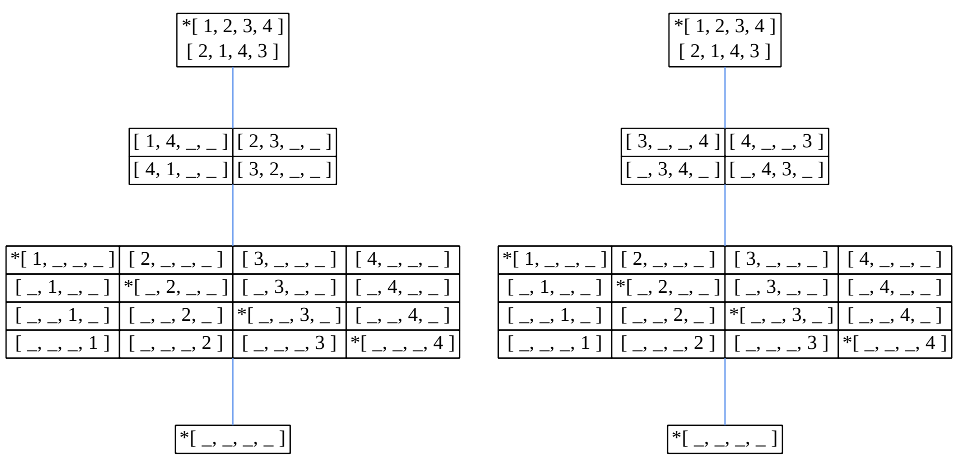

FIGURE 5. Eggbox diagram of $\mathrm{S}(\mathscr{X})$ and $\mathrm{S}(\mathscr{Y})$.

resent $\mathcal{D}$-classes, rows represent $\mathcal{R}$-classes, and columns represent $\mathcal{L}$-classes. The stars indicate idempotent elements. The lines linking $\mathcal{D}$-classes represent consecutive $\mathcal{D}$-classes in the $\leq_{\mathcal{J}}$-relation. The symbol $[q_1, q_2, q_3, q_4]$ represents an element s of $S(\mathscr{X})$ as a partial transformation of $Q(\mathscr{X}) \setminus \{\emptyset\}$, with $q_j = \_$ if $js = \emptyset$, and $q_j = js$ otherwise, for $1 \leq j \leq 4$. Note that the

flow equivalence invariants $\mathfrak{FD}\big(S(\mathscr{X})\big)$ and $\mathfrak{FD}\big(S(\mathscr{Y})\big)$ are equal. We turn our attention to $\mathfrak{KD}\big(S(\mathscr{X})\big)$ and $\mathfrak{KD}\big(S(\mathscr{Y})\big)$. Figure 6 depicts the Krieger covers of $\mathscr{X}$ and $\mathscr{Y}$, respectively. Note in particular they have isomorphic proper communication graphs. The identity element a^2 has rank 7 in $\mathfrak{K}(\mathscr{X})$

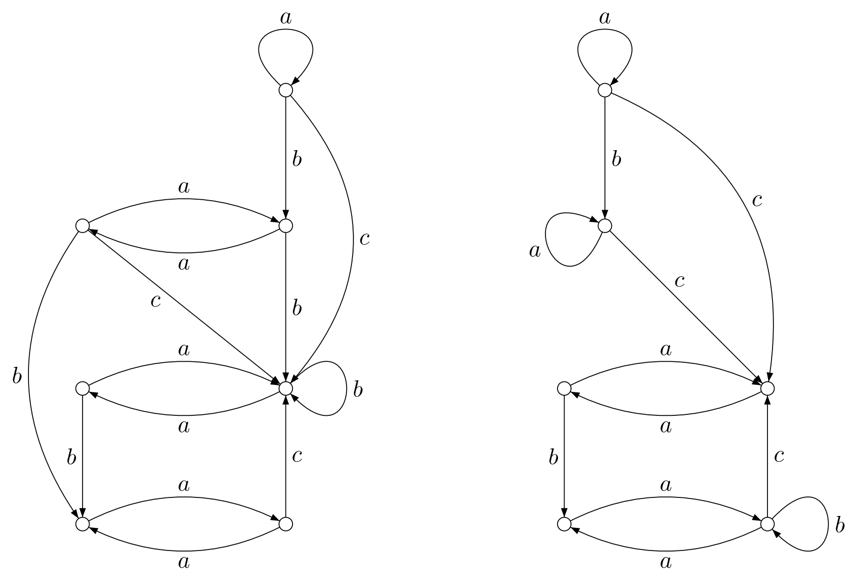

FIGURE 6. Krieger covers of the shifts in Example 6.7.

and rank 6 in $\mathfrak{K}(\mathscr{Y})$. Therefore, $\mathfrak{KD}\big(S(\mathscr{X})\big) \neq \mathfrak{KD}\big(S(\mathscr{Y})\big)$, because they do not have the same label at the $\mathcal{D}$-class of the identity, at the top of the Hasse diagram. Hence, $\mathscr{X}$ and $\mathscr{Y}$ are not flow equivalent, by Theorem 6.3.

In Example 6.7 we used a general fact concerning sofic shifts whose syntactic semigroup is a monoid: since the rank of the identity is invariant under flow equivalence (by Theorem 6.3), it follows that flow equivalent shifts whose syntactic semigroup is a monoid have the same number of vertices in the Krieger cover, and also in the Fischer cover if they are synchronizing.

Clearly, two equivalent categories have the same local monoids, up to isomorphism. Therefore, from Theorem 4.2, we immediately extract the following criterion for sofic shifts whose syntactic semigroup is a monoid.

Proposition 6.8. *Let $\mathscr{X}$ and $\mathscr{Y}$ be flow equivalent shifts. If $S(\mathscr{X})$ is a monoid then $S(\mathscr{X})$ embeds into $S(\mathscr{Y})$. In particular, if $S(\mathscr{X})$ and $S(\mathscr{Y})$ are both finite monoids, then $S(\mathscr{X})$ and $S(\mathscr{Y})$ are isomorphic.* □

In fact, it is known that if M is a monoid and N is a semigroup with local units, then M is Morita equivalent to N if and only if there is an idempotent $e \in N$ with $N = NeN$ and $M \cong eMe$ [70].

Example 6.9. We return to the shifts $\mathscr{X}$ and $\mathscr{Y}$ from Example 6.7. We claim that the monoids $S(\mathscr{X})$ and $S(\mathscr{Y})$ are not isomorphic, which enables us to

apply Proposition 6.8 to show again that $\mathscr{X}$ and $\mathscr{Y}$ are not flow equivalent. Let $D_{\mathscr{X}}$ and $D_{\mathscr{Y}}$ be the unique non-regular $\mathcal{D}$-classes of $S(\mathscr{X})$ and $S(\mathscr{Y})$, respectively (see Figure 5). Then the set $\{d^2 \colon d \in D_{\mathscr{X}}\}$ is the $\mathcal{R}$-class of $[1,\_,\_,\_]$ in $S(\mathscr{X})$, while $\{d^2 \colon d \in D_{\mathscr{Y}}\}$ contains pairs of elements which are not $\mathcal{R}$-equivalent in $S(\mathscr{Y})$, such as $[3,\_,\_,\_]$ and $[\_,\_,\_,3]$. This proves the claim.

Next we give an example concerning non-sofic synchronizing shifts.

Example 6.10. The word bc is synchronizing for the two labeled graphs in Figure 7; these labeled graphs are the Fischer covers of synchronizing shifts, denoted respectively $\mathscr{X}$ and $\mathscr{Y}$, over the alphabet $\{a, b, c\}$ (cf. Theorem 3.9). If $u \in L(\mathscr{X})$, then the action of $\delta_{\mathscr{X}}(u)$ in $\mathfrak{F}(\mathscr{X})$ has rank one or infinite

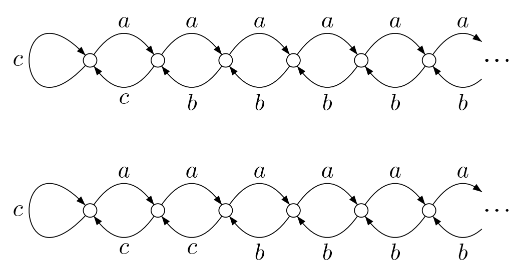

Figure 7. Fischer covers of two synchronizing shifts.

rank, depending on whether c is a factor of u or not. On the other hand, the idempotent $\delta_{\mathscr{Y}}(ac)$ has rank two. Therefore $\mathscr{X}$ and $\mathscr{Y}$ are not flow equivalent, by Theorem 6.3, since $\mathfrak{F}\mathfrak{D}(\mathscr{X}) \neq \mathfrak{F}\mathfrak{D}(\mathscr{Y})$.

7. Classes of sofic shifts defined by pseudovarieties of semigroups

A *pseudovariety of semigroups* is a class of finite semigroups closed under the formation of finite direct products, subsemigroups and homomorphic images. This notion is a fundamental one in finite semigroup theory, as pseudovarieties are considered to provide a correct framework to study finite semigroups, as well their connections to other subjects, such as formal language theory [24, 65, 68].

Given a pseudovariety of finite semigroups V, we denote by $\mathcal{S}(\mathsf{V})$ the class of shifts $\mathscr{X}$, necessarily sofic, such that $S(\mathscr{X})$ belongs to V. Since the two-element monoid $U_1 = \{0, 1\}$, with the usual multiplication, is a subsemigroup of $S(\mathscr{X})$ for every sofic shift, the class $\mathcal{S}(\mathsf{W})$ is non-empty if and only if the pseudovariety W contains the pseudovariety Sl generated by U_1 (alternatively, Sl is the pseudovariety of *semilattices*, that is, of commutative semigroups all of whose elements are idempotents).

7.1. **The localization of a pseudovariety.** For a pseudovariety of finite semigroups V, the *localization of* V, denoted by $\mathcal{L}\mathsf{V}$, is the pseudovariety of finite semigroups S whose local monoids eSe (with $e \in E(S)$) belong to V. We mention that it follows easily from known results that an irreducible shift $\mathscr{X}$ is of finite type if and only if $S(\mathscr{X}) \in \mathcal{L}\mathsf{SI}$ (cf. [18, Proposition 4.2]).

Theorem 7.1. *If* V *is a pseudovariety of semigroups then the class* $\mathcal{S}(\mathcal{L}\mathsf{V})$ *is closed under flow equivalence.*

Proof. Suppose that the sofic shifts $\mathscr{X}$ and $\mathscr{Y}$ are flow equivalent. By Theorem 4.2, the semigroups $S(\mathscr{X})$ and $S(\mathscr{Y})$ have the same local monoids, up to isomorphism. Hence $S(\mathscr{X}) \in \mathcal{L}\mathsf{V}$ if and only if $S(\mathscr{Y}) \in \mathcal{L}\mathsf{V}$. □

Theorem 7.1 provides a method to show that some natural classes of sofic shifts are closed under flow equivalence. As an example, let us consider the class of *almost finite type* shifts. For background and motivation see [52, Section 13.1]. A comprehensive list of characterizations of almost finite shifts can also be found in [3]. The shifts in Examples 6.4 and 6.7 are almost finite type shifts. Denote by $\mathcal{S}_I(\mathsf{V})$ the intersection of $\mathcal{S}(\mathsf{V})$ with the class of irreducible shifts. It turns out that the class of almost finite type shifts is $S_I(\mathcal{L}\,\mathsf{ECom})$, where ECom is the pseudovariety of semigroups whose idempotents commute [18]. Hence, from Theorem 7.1 we deduce the following result, which was first proved in [28].

Theorem 7.2. *The class of almost finite type shifts is closed under flow equivalence.* □

We give yet another example. In [4,7] a class of sofic shifts called *aperiodic shifts* is studied. In [7] it is proved that the class of irreducible aperiodic shifts is the class $S_I(\mathsf{A})$, where A is the pseudovariety of *aperiodic semigroups* (a semigroup is aperiodic if the $\mathcal{H}$-relation is trivial), and that this class is closed under conjugacy. Since $\mathsf{A} = \mathcal{L}\mathsf{A}$, from Theorem 7.5 we obtain the following improvement.

Theorem 7.3. *The class of irreducible aperiodic shifts is closed under flow equivalence.* □

7.2. **The divisional equivalence class of $\mathbb{K}(\mathscr{X})$.** A category C *divides* a category D if there are functors $F\colon U \longrightarrow D$ and $G\colon U \longrightarrow C$ such that F is faithful and G is quotient (i.e., is bijective on objects and full). The relation of division is a preorder. See [68], or [71], the latter being where this notion was for the first time systematically applied in finite semigroup theory. Equivalent categories are divisionally equivalent, hence we have the following simple corollary of Theorem 4.2.

Proposition 7.4. *The divisional equivalence class of* $\mathbb{K}(\mathscr{X})$ *is a flow equivalence invariant.* □

Note that we did not assume that $\mathscr{X}$ is sofic. We briefly describe an application of Proposition 7.4 for sofic shifts. The *semidirect product* of pseudovarieties, denoted by $\mathsf{V} * \mathsf{W}$, is the pseudovariety generated by semidirect

products of elements of V with elements of W, in this order. We refer the reader interested in further details to [68]. Let D be the pseudovariety of finite semigroups whose idempotents are right zeroes. Tilson's Delay Theorem [71] states that a finite semigroup S belongs to $\mathsf{V} * \mathsf{D}$ if and only if $\mathbb{K}(S)$ possesses a certain technical property[3] related with V which is invariant under division of categories. Since the divisional equivalence class of $\mathbb{K}(\mathscr{X})$ is a flow invariant, from Tilson's Delay Theorem we get the following.

Theorem 7.5. *If V is a pseudovariety of semigroups then the class $\mathcal{S}(\mathsf{V} * \mathsf{D})$ is closed under flow equivalence.* □

That $\mathcal{S}(\mathsf{V} * \mathsf{D})$ is closed under conjugacy when $\mathsf{Sl} \subseteq \mathsf{V}$ was also proved in [18]. Conversely, if $\mathcal{S}(\mathsf{V})$ is non-empty and closed under conjugacy, then $\mathsf{Sl} * \mathsf{D} \subseteq \mathsf{V}$ and $\mathcal{S}(\mathsf{V}) = \mathcal{S}(\mathsf{V} * \mathsf{D})$ [18].

In [18] one finds examples of non-empty classes of the form $\mathcal{S}(\mathsf{V} * \mathsf{D})$ which are not of the form $\mathcal{S}(\mathcal{L}\mathsf{V})$. On the other hand, we mention that one has $\mathcal{L}\mathsf{V} = \mathcal{L}\mathsf{V} * \mathsf{D}$ (see for example [1, Proposition 10.6.13]), and so the classes in Theorem 7.1 are special cases of those in Theorem 7.5.

8. Subsynchronizing subshifts of a sofic shift

As a further application of our main results, we apply them to the poset of subsynchronizing subshifts of a sofic shift considered in [40]. This poset, whose definition is recalled in this section, provides information about the structure of a reducible sofic shift.

Let $\mathscr{X}$ be a sofic subshift of $A^{\mathbb{Z}}$. We recall some definitions and remarks from [40]. If m is a synchronizing word for $\mathscr{X}$, then m is *magic* for $\mathscr{X}$ if $mum \in L(\mathscr{X})$ for some $u \in A^*$. If m is magic for $\mathscr{X}$, then the set

$$\{v \in A^+ \mid \exists x \in A^* : mxv \in L(\mathscr{X})\}$$

is the set of finite blocks of a sofic subshift of $\mathscr{X}$. This shift is denoted $S(m)$. If M is a set of magic words for $\mathscr{X}$, then $S(M)$ denotes the sofic shift $\bigcup_{m \in M} S(m)$. A subshift of $\mathscr{X}$ of the form $S(M)$ is called a *subsynchronizing subshift of $\mathscr{X}$*. The set $\mathsf{Subs}(\mathscr{X})$ of subsynchronizing subshifts of $\mathscr{X}$ is finite; see Lemma 8.1 below. It may be empty. If $\mathscr{X}$ is irreducible, then $\mathsf{Subs}(\mathscr{X}) = \{\mathscr{X}\}$.

Let $\mathscr{X}$ be a sofic shift. Say that $s \in S(\mathscr{X})$ is *synchronizing* if $s = \delta_{\mathscr{X}}(u)$ with u synchronizing. Note that s is synchronizing if and only if $rs, st \neq 0$ implies $rst \neq 0$, for all $r, t \in S(\mathscr{X})$. It follows from Lemma 3.1 that the synchronizing elements of $S(\mathscr{X})$, together with 0, form an ideal and that a synchronizing element s is idempotent if and only if $s^2 \neq 0$.

A synchronizing idempotent of $S(\mathscr{X}) \setminus \{0\}$ will be called a *magic idempotent* for $\mathscr{X}$. Let e be a magic idempotent for $\mathscr{X}$, and let u be a word such that $e = \delta_{\mathscr{X}}(u)$. Then u is a magic word for $\mathscr{X}$. Clearly, if $\delta_{\mathscr{X}}(u) = \delta_{\mathscr{X}}(v)$,

[3]Namely, a finite semigroup S belongs to $\mathsf{V} * \mathsf{D}$ if and only if $\mathbb{K}(S)$ *divides* an element of V; the technicality amounts to define the meaning of a category dividing a semigroup.

then $S(u) = S(v)$. We may then define $S(e)$ as being $S(u)$. If M is a set of magic idempotents for $\mathscr{X}$, then $S(M)$ denotes the sofic shift $\bigcup_{e\in M} S(e)$.

In the proof of the following lemma, $\mathrm{Fac}(X)$ denotes the set of non-empty words which are factors of a language X.

Lemma 8.1. *If m is a magic word for $\mathscr{X}$, then $S(m) = S(e)$ for some magic idempotent e for $\mathscr{X}$.*

Proof. Since m is a magic word for $\mathscr{X}$, there is $u \in L(\mathscr{X})$ such that $mum \in L(\mathscr{X})$. We claim that $e = \delta_{\mathscr{X}}(mu)$ is a magic idempotent for $\mathscr{X}$ such that $S(m) = S(e)$.

We first show that e is an idempotent. It follows from Lemma 3.1 that we must show $(mu)^2 \in L(\mathscr{X})$. But $mum, mu \in L(\mathscr{X})$ implies $mumu \in L(\mathscr{X})$. Also mu is synchronizing by Lemma 3.1. Thus e is a magic idempotent.

We have $L(S(m)) = \mathrm{Fac}(R_{\mathscr{X}}(m))$ and $L(S(e)) = \mathrm{Fac}(R_{\mathscr{X}}(mu))$ by definition. Clearly, $\mathrm{Fac}(R_{\mathscr{X}}(mum)) \subseteq \mathrm{Fac}(R_{\mathscr{X}}(mu)) \subseteq \mathrm{Fac}(R_{\mathscr{X}}(m))$. But $R_{\mathscr{X}}(mum) = R_{\mathscr{X}}(m)$ by Lemma 3.1. This shows that $L(S(m)) = L(S(e))$, thus $S(m) = S(e)$. □

Next we show that a regular $\mathcal{D}$-class of $S(\mathscr{X})$ containing magic idempotents determines a subsynchronizing subshift of $\mathscr{X}$.

Lemma 8.2. *If e, f are magic idempotents for $\mathscr{X}$ such that $e \mathrel{\mathcal{D}} f$, then $S(e) = S(f)$.*

Proof. Since $e \mathrel{\mathcal{D}} f$, there are $u, v \in A^+$ such that $e = \delta_{\mathscr{X}}(uv)$ and $f = \delta_{\mathscr{X}}(vu)$. Then we have

$$L(S(e)) = \mathrm{Fac}(R_{\mathscr{X}}(uvuv)) \subseteq \mathrm{Fac}(R_{\mathscr{X}}(vu)) = L(S(f)),$$

whence $S(e) \subseteq S(f)$. By symmetry, we obtain $S(e) = S(f)$. □

The following stream of results concerns the relationship between $\mathsf{Subs}(\mathscr{X})$ and $\mathsf{Subs}(\mathscr{Y})$ when $\mathbb{K}(\mathscr{X})$ are $\mathbb{K}(\mathscr{Y})$ equivalent. In some of the proofs, we shall use the following property of a sofic shift $\mathscr{X}$, which is a consequence of every sofic shift being presented by a finite essential graph: if $s \in S(\mathscr{X}) \setminus \{0\}$, then there are $r, t \in S(\mathscr{X})$ and idempotents $e, f \in S(\mathscr{X})$ with $erstf \neq 0$.

Lemma 8.3. *Let $F\colon \mathbb{K}(\mathscr{X}) \longrightarrow \mathbb{K}(\mathscr{Y})$ be an equivalence, where $\mathscr{X}$ and $\mathscr{Y}$ are sofic shifts. If e is a magic idempotent for $\mathscr{X}$, then $F(e)$ is a magic idempotent for $\mathscr{Y}$.*

Proof. Let s, t be elements of $S(\mathscr{Y})$ such that $sF(e) \neq 0$ and $F(e)t \neq 0$. Since $\mathscr{Y}$ is sofic and F is essentially surjective, there are $s', t' \in S(\mathscr{Y})$ and idempotents g and h of $S(\mathscr{X})$ such that $F(g)s'sF(e) \neq 0$ and $F(e)tt'F(h) \neq 0$. The fact that F is an equivalence guarantees that there are morphisms (g, u, e) and (e, v, h) of $\mathbb{K}(\mathscr{Y})$ such that

$$F(g, u, e) = (F(g), F(g)s'sF(e), F(e)),$$
$$F(e, v, h) = (F(e), F(e)tt'F(h), F(h)),$$

and $u \neq 0$, $v \neq 0$ (recall Remark 2.1). Because e is synchronizing, we have $uv = uev \neq 0$, and so $F(g, uv, h) \neq (F(g), 0, F(h))$ (again by Remark 2.1). Since

$$F(g, uv, h) = (F(g), F(g)s'sF(e)tt'F(h), F(h)),$$

we then have $sF(e)t \neq 0$, showing that $F(e)$ is a magic idempotent. □

Lemma 8.3 guarantees that, in the following lemma, the set $S(GF(M))$ is well defined.

Lemma 8.4. *Let $F\colon \mathbb{K}(\mathscr{X}) \longrightarrow \mathbb{K}(\mathscr{Y})$ be an equivalence, where $\mathscr{X}$ and $\mathscr{Y}$ are sofic shifts. If M is a set of magic idempotents for $\mathbb{K}(\mathscr{X})$ and G is a quasi-inverse of F, then $S(M) = S(GF(M))$.*

Proof. It suffices to show that $S(e) = S(GF(e))$, when e is a magic idempotent for $\mathscr{X}$. Since $e \,\mathcal{D}\, GF(e)$, the result follows from Lemma 8.2. □

In the following proposition we see what is the effect of an equivalence of Karoubi envelopes on the poset $\mathsf{Subs}(\mathscr{X})$.

Proposition 8.5. *Let $F\colon \mathbb{K}(\mathscr{X}) \longrightarrow \mathbb{K}(\mathscr{Y})$ be an equivalence, where $\mathscr{X}$ and $\mathscr{Y}$ are sofic shifts. Suppose M_1 and M_2 are sets of magic idempotents for $\mathscr{X}$. Then $S(M_1) \subseteq S(M_2)$ if and only if $S(F(M_1)) \subseteq S(F(M_2))$.*

Proof. Suppose first that $S(F(M_1)) \subseteq S(F(M_2))$. Take $u \in L(S(M_1))$. Let $s = \delta_{\mathscr{X}}(u)$. Then, there are $e \in M_1$ and $t \in S(\mathscr{X})$ such that $ets \neq 0$. Moreover, since $\mathscr{X}$ is sofic, there are $t', f \in S(\mathscr{X})$ such that $etst'f \neq 0$ and f is idempotent. Therefore, $(e, etst'f, f)$ is a morphism of $\mathbb{K}(\mathscr{X})$ distinct from $(e, 0, f)$. Let $(F(e), z, F(f))$ be its image under F. Because F is an equivalence, it follows from Remark 2.1 that $z \neq 0$. Since $F(e)z \neq 0$, we have $\delta_{\mathscr{Y}}^{-1}(z) \subseteq L(S(F(e)))$. As $S(F(e)) \subseteq S(F(M_1)) \subseteq S(F(M_2))$, there are $g \in M_2$ and $r \in S(\mathscr{Y})$ such that $r = F(g)rF(e)$ and $rz \neq 0$. Since F is an equivalence, there is a morphism $(g, r', e) \in \mathbb{K}(\mathscr{X})$ such that $F(g, r', e) = (F(g), r, F(e))$. Note that $F((g, r', e)(e, etst'f, f)) = (F(g), rz, F(f))$. Therefore, we have $gr'ets \neq 0$ by Remark 2.1, and so $u \in L(S(g)) \subseteq L(S(M_2))$. This shows that $L(S(M_1)) \subseteq L(S(M_2))$, that is, $S(M_1) \subseteq S(M_2)$.

Conversely, suppose that $S(M_1) \subseteq S(M_2)$. Let G be a quasi-inverse of F. By Lemma 8.4, we have $S(M_i) = S(GF(M_i))$, for $i = 1, 2$. Applying to G the “if” part of the proposition, proved in the previous paragraph, we obtain $S(F(M_1)) \subseteq S(F(M_2))$. □

Corollary 8.6. *Let $F\colon \mathbb{K}(\mathscr{X}) \longrightarrow \mathbb{K}(\mathscr{Y})$ be an equivalence, where $\mathscr{X}$ and $\mathscr{Y}$ are sofic shifts. Then the mapping $\Psi_F\colon \mathsf{Subs}(\mathscr{X}) \longrightarrow \mathsf{Subs}(\mathscr{Y})$ defined by $\Psi_F(S(M)) = S(F(M))$, where M runs over the sets of magic idempotents, is a well-defined isomorphism of posets. If G is a quasi-inverse of F, then Ψ_G is the inverse of Ψ_F.*

Proof. It follows immediately from Proposition 8.5 that Ψ_F is a well-defined and order-preserving function. Moreover, we conclude from Lemma 8.4 that Ψ_F and Ψ_G are mutually inverse. □

As a direct consequence of Theorem 4.2 and Corollary 8.6, we deduce the following.

Corollary 8.7. *The order structure of the poset of subsynchronizing subshifts of a sofic shift is invariant under flow equivalence.* □

The invariance of the order structure of $\mathsf{Subs}(\mathscr{X})$ under conjugacy of sofic shifts was proved in [40]. Several concrete examples were examined in that paper. Actually a more general result concerning conjugacy was obtained in [40] (although not used in the examples): viewing $\mathsf{Subs}(\mathscr{X})$ as a labeled poset, where the label of each element is its conjugacy class, one obtains a conjugacy invariant. We shall not give a new proof of this fact using our methods, since we would not obtain a significative simplification. However, we do generalize it to flow equivalence, in the next theorem. For a sofic shift $\mathscr{X}$, its *labeled flow poset of subsynchronizing subshifts* is the labeled poset obtained from $\mathsf{Subs}(\mathscr{X})$ in which the label of each element of $\mathsf{Subs}(\mathscr{X})$ is its flow equivalence class.

Theorem 8.8. *The labeled flow poset of subsynchronizing subshifts of a sofic shift is a flow equivalence invariant.*

Proof. In view of the proof of Corollary 8.7 and the results from [40], it suffices to show that if $\mathscr{X}'$ is the symbol expansion of $\mathscr{X}$ relatively to a letter α, then there is an equivalence $F\colon \mathbb{K}(\mathscr{X}) \longrightarrow \mathbb{K}(\mathscr{X}')$ such that $S(F(e))$ is the symbol expansion of $S(e)$ relatively to α, for every magic idempotent e for $\mathscr{X}$. This is done in an appendix, in Proposition B.1, using some tools introduced in Section 12. □

9. Markov-Dyck shifts

The examples we have given so far were all of sofic or synchronizing shifts. In this section we apply Theorem 4.2 to classify with respect to flow equivalence a class of shifts, introduced and studied by W. Krieger and his collaborators [32, 44, 46–48] that in general are non-sofic and non-synchronizing. These shifts, called *Markov-Dyck shifts*, are the main subject of the article [48]. Previously, they had appeared as special cases of more general constructions in [32, 44, 46, 47]. These shifts are defined in terms of graph inverse semigroups.

An inverse semigroup S is a semigroup such that for all $s \in S$, there is a unique element s^* such that both $ss^*s = s$ and $s^*ss^* = s^*$ hold. Note that ss^*, s^*s are then both idempotents. Idempotents of an inverse semigroup commute and hence form a subsemigroup which is a semilattice, with ordering given by $e \leq f$ if and only if $ef = e$. In an inverse semigroup, one has that $s \mathrel{\mathcal{L}} t$ (respectively, $s \mathrel{\mathcal{R}} t$) if and only if $ss^* = tt^*$ (respectively, $s^*s = t^*t$). Consequently, one has $ss^* \mathrel{\mathcal{D}} s^*s$. See [50, 63] for more on inverse semigroups.

Graph inverse semigroups, in the special case of graphs with no multiple-edges, were first considered by Ash and Hall [2]. They were considered in greater generality in [63, 64]. The recent paper [38] is a useful reference.

Let G be a (directed) graph and let G^* be the free category generated by G. The objects of G^* are the vertices of G and the morphisms are (directed) paths, including an empty path at each vertex. Since we are adopting the Category Theory convention for the composition of morphisms, in this context a path of G should be understood as follows: a non-empty finite sequence of edges $x_1 \cdots x_n$ is a path of G if the domain of x_i is the co-domain of x_{i+1}, for $1 \leq i < n$. Of course, the empty path at a vertex q, denoted 1_q, is the identity of G^* at q. The domain and co-domain of a path u will be denoted respectively by αu and ωu.

Associated to G is an inverse semigroup P_G, called the *graph inverse semigroup of* G. The underlying set of P_G is the set of all pairs (x, y) of morphism $x, y \in G^*$ such that x and y have a common domain, together with an extra element 0, which is the zero element of P_G. The pair (x, y) is usually denoted xy^{-1}. We think on y^{-1} as a formal inverse of the path y, obtained by reversing the directions of the edges in y. We make the identifications $y = (y, 1_{\alpha y})$ and $y^{-1} = (1_{\alpha y}, y)$. Moreover, we have $1_q^{-1} = 1_q$, for every vertex q of G. The multiplication between elements of $P_G \setminus \{0\}$ is given by the following rules:

$$xy^{-1} \cdot uv^{-1} = \begin{cases} xzv^{-1} & \text{if } u = yz \text{ for some path } z, \\ x(vz)^{-1} & \text{if } y = uz \text{ for some path } z, \\ 0 & \text{otherwise.} \end{cases}$$

The semigroup P_G is an inverse semigroup in which $(xy^{-1})^* = yx^{-1}$ and $0^* = 0$. Its non-zero idempotents are the pairs of the form xx^{-1}, with $x \in G^*$. Note that 1_q is idempotent, for every vertex q of G, and $x^{-1}x = 1_{\alpha x}$ for every $x \in G^*$.

If the out-degree of each vertex is at least 1, then P_G is generated as a semigroup with zero by the set Σ_G of elements of the form x or x^{-1}, with x an edge of G. (Note that if q is a vertex and e is an out-going edge from q, then $1_q = e^{-1}e$.) We may therefore consider the unique homomorphism $\pi\colon (\Sigma_G)_0^+ \longrightarrow P_G$ of semigroups with zero extending the identity map on Σ_G. The language $\pi^{-1}(P_G \setminus \{0\})$ is clearly factorial. It is prolongable because in P_G we have $x = xx^{-1}x$ and $x^{-1} = x^{-1}xx^{-1}$, for every edge x of G.

From now on we assume that the graph G, and hence Σ_G, is finite. The *Markov-Dyck shift* associated to the graph G is the subshift D_G of $(\Sigma_G)^{\mathbb{Z}}$ such that $L(D_G) = \pi^{-1}(P_G \setminus \{0\})$. In particular, the syntactic semigroup of D_G is a non-trivial homomorphic image of P_G.

Let us now characterize when P_G is the syntactic semigroup of D_G. To do this, we shall need a result about inverse semigroups that undoubtedly is known to experts, but we cannot find it explicitly in the literature. Let us call a congruence trivial if it is the equality relation. An inverse semigroup

S is called *fundamental* if every non-trivial congruence identifies some pair of idempotents. A semilattice E with zero is called 0-*disjunctive* if for all $0 < e < f$, there exists $0 < e' < f$ such that $ee' = 0$.

Lemma 9.1. *Let S be an inverse semigroup with zero. Then the following are equivalent:*

1. *there is no non-trivial congruence on S separating $S \setminus \{0\}$ and $\{0\}$ (i.e., S is the syntactic semigroup of $S \setminus \{0\}$);*
2. *S is fundamental and $E(S)$ is 0-disjunctive.*

Proof. Assume the first condition and suppose that $\equiv$ is a non-trivial congruence on S. Then $s \equiv 0$ for some $s \neq 0$ by hypothesis and hence $ss^* \equiv 0s^* = 0$. Thus $\equiv$ does not separate idempotents. It follows that S is fundamental.

To see that $E(S)$ is 0-disjunctive, for $s \in S$, let $c(s) = \{(u, v) \in S^1 \times S^1 \mid usv = 0\}$ where S^1 denotes S with an adjoined identity. Then the equivalence relation $\equiv$ such that $s \equiv t \Leftrightarrow c(s) = c(t)$ is a congruence separating $S \setminus \{0\}$ and $\{0\}$ (it is the syntactic congruence of the subset $\{0\}$). Thus by hypothesis, $\equiv$ is trivial. Suppose $0 < e < f$ are idempotents. Then $c(e) \neq c(f)$. Note that $ufv = 0$ implies $uev = uefv = uu^*uefv = ueu^*ufv = 0$ and so $c(f) \subseteq c(e)$. Thus we can find $u, v \in S^1$ with $uev = 0$, but $ufv \neq 0$. Let $e' = u^*ufvv^* \in E(S)$. Note that $e' \leq f$ and $ue'v = ufv \neq 0$. Thus $e' \neq 0$. Moreover, $ee' = u^*uevv^* = 0$, and so in particular $e' \neq f$ as $0 < e = ef$. Therefore $0 < e' < f$ and $ee' = 0$ and so we have established that $E(S)$ is 0-disjunctive.

For the converse, assume the second condition and let $\equiv$ be a non-trivial congruence on S. We must show that $\equiv$ does not separate $S \setminus \{0\}$ from $\{0\}$. Because S is fundamental, we can find idempotents $e \neq f$ with $e \equiv f$. If either e or f is 0, we are done and so we may assume $e \neq 0 \neq f$. If $ef = 0$, then $ef \equiv f$ and again we are done. If $ef = f$, then $0 < f < e$, and so, as $E(S)$ is 0-disjunctive, there is and idempotent f' such that $0 < f' < e$ and $ff' = 0$. Then $f' = ef' \equiv ff' = 0$ and we are done. So we may assume that $0 < ef < f$, which is the last case to be considered. Then, as $E(S)$ is 0-disjunctive, we may find and idempotent e' with $0 < e' < f$ and $efe' = 0$. Then $0 = efe' \equiv fe' = e'$ and so again, we are done. This completes the proof. □

Remark 9.2. Notice that this lemma implies the well known result that an inverse semigroup S is congruence-free if and only if it is fundamental, 0-simple and $E(S)$ is 0-disjunctive because for a 0-simple semigroup any congruence which does not separate 0 from $S \setminus \{0\}$ must collapse all of S to 0.

Lemma 9.3. *Let G be a finite graph such that each vertex has out-degree at least one. Then P_G is the syntactic semigroup of D_G if and only if G has no vertex of in-degree exactly one.*

Proof. Clearly, P_G is the syntactic semigroup of D_G if and only if each non-trivial congruence on P_G fails to separate $P_G \setminus \{0\}$ from $\{0\}$. By Lemma 9.1, this occurs if and only if P_G is fundamental and $E(P_G)$ is 0-disjunctive. The inverse semigroup P_G is always aperiodic and hence fundamental (cf. [38], where aperiodic semigroups are called *combinatorial*). On the other hand, it is shown in [38, Lemma 2.9] that $E(P_G)$ is 0-disjunctive if and only if G has no vertex of in-degree exactly one. This completes the proof. $\square$

It is easy to see that P_G is finite if and only if G is acyclic. Therefore, the above lemma implies that the shift D_G is not sofic in general.

In view of our mains results and of Lemma 9.3, we are naturally interested in investigating the Karoubi envelope of P_G.

Recall that a morphism $f\colon d \longrightarrow c$ of a category C is said to be a *split monomorphism* if it has a left inverse, that is, if there is a morphism $g\colon c \longrightarrow d$ such that $gf = 1_d$. The composition of two split monomorphisms is a split monomorphism, and so we can consider the subcategory L_C of C formed by the split monomorphisms of C. Note that an equivalence $C \longrightarrow D$ restricts to an equivalence $L_C \longrightarrow L_D$.

For a semigroup S, a morphism (e, s, f) of $\mathbb{K}(S)$ is a split monomorphism if and only if $s \mathrel{\mathcal{L}} f$ [55]. In particular, if S has a zero, then the only split monomorphisms (e, s, f) with $s = 0$ are of the form $(e, 0, 0)$. In other words, if we consider the full subcategory $\mathbb{L}(S)$ of $L_{\mathbb{K}(S)}$ whose objects are the non-zero idempotents, then there are no morphisms of the form $(e, 0, f)$ in this subcategory. Note that an equivalence $L_{\mathbb{K}(S)} \longrightarrow L_{\mathbb{K}(T)}$ restricts to an equivalence $\mathbb{L}(S) \longrightarrow \mathbb{L}(T)$ because 0 is the unique initial object of $L_{\mathbb{K}(S)}$ and similarly for $L_{\mathbb{K}(T)}$.

The argument for the following lemma is essentially contained in [69]

Lemma 9.4. *For every graph G, the category $\mathbb{L}(P_G)$ is equivalent to the free category G^*.*

Proof. An object of $\mathbb{L}(P_G)$ is an idempotent of the form uu^{-1}, with $u \in G^*$. Since $uu^{-1} \mathrel{\mathcal{D}} 1_{\alpha u}$, the category $\mathbb{L}(P_G)$ is equivalent to the full subcategory $\mathbb{L}(P_G)'$ whose objects are the idempotents of the form 1_q. If $u, v \in G^*$ are such that $(1_q, uv^{-1}, 1_r)$ is a split monomorphism of $\mathbb{K}(P_G)$, then $1_r = xy^{-1}uv^{-1}$ for some $x, y \in G^*$, and so $v = 1_r$ by the rules defining the multiplication in G^*. Therefore, $\mathbb{L}(P_G)'$ is isomorphic to G^*. $\square$

It is well known that two free categories G^* and H^* on graphs G, H are equivalent if and only if G and H are isomorphic. Since we don't know a precise reference, we sketch a proof.

Lemma 9.5. *Suppose that G, H are graphs with G^* equivalent to H^*. Then G and H are isomorphic.*

Proof. In a free category, there are no isomorphisms other than the identities. Hence, two functors with codomain a free category are isomorphic if and only if they are equal. It follows that any equivalence $F\colon G^* \longrightarrow H^*$ is actually

an isomorphism. A morphism u of a free category is an edge if and only if it cannot be factored $u = vw$ with v, w non-empty paths. Thus F must restrict to a graph isomorphism $G \longrightarrow H$. □

As a corollary of the preceding two lemmas, we deduce that Morita equivalent graph inverse semigroups have isomorphic underlying graphs.

Corollary 9.6. *Let G, H be graphs. Then P_G is Morita equivalent to P_H if and only if G and H are isomorphic.*

The case where G is a finite one-vertex graph with at least 2 loops has received special attention in the literature and motivates the general case. If $N \geq 2$ is the number of loops of the one-vertex graph G, then the corresponding Markov-Dyck shift is denoted D_N, and is called the *Dyck shift of rank N*. In [58] it is proved, by means of the computation of certain flow invariant abelian groups, that if D_N and D_M are flow equivalent then $N = M$. This is proved here again, as a special case of the next result.

Theorem 9.7. *Let G and H be finite graphs such that the out-degree of each vertex of G and H is at least one and the in-degree of each vertex of G and H is not one. Then D_G and D_H are flow equivalent if and only if G and H are isomorphic.*

Proof. Sufficiency is obvious. Conversely, suppose that D_G and D_H are flow equivalent. By Lemma 9.3, P_G and P_H are the syntactic semigroups of D_G and D_H, respectively. They are inverse semigroups and hence have local units. Theorem 4.2 implies that P_G and P_H are Morita equivalent. Corollary 9.6 then yields $G \cong H$. □

Since on the one hand conjugacy of shifts implies flow equivalence of shifts, and on the other if $G \cong H$, then D_G and D_H are obviously conjugate, Theorem 9.7 also yields a classification theorem for conjugacy between Markov-Dyck shifts. Namely, D_G and D_H are conjugate if and only if G and H are isomorphic (under the hypotheses of the theorem). This generalizes results of Krieger [46, 47], who showed that if G and H are strongly connected and each vertex of G and H has in-degree at least 2, then D_G is conjugate to D_H if and only if G and H are isomorphic. On one hand, Krieger proved in [47] that if G and H are finite graphs such that P_G and P_H are isomorphic, then G and H are isomorphic. On the other hand, he used a property of shifts, introduced by him [46], called *Property (A)*, which is a conjugacy invariant. To shifts with such a property, he associated a semigroup whose isomorphism class is a conjugacy invariant. It turns out that D_G is a Property (A) shift if G is strongly connected [33]. In [33] it is given a condition under which P_G is the semigroup associated to a Property (A) shift. This condition is satisfied if G is strongly connected and each vertex of G has in-degree at least two, in which case P_G is actually associated to D_G (cf. [46] and the paragraph before [32, Lemma 4.1]). Also in [33], it is shown that the conjugacy class of a Markov-Dyck shift defined

by a strongly connected graph with at most three vertices is determined by the isomorphism class of the graph. In view of these results, it is natural to make the question (which is left open) of whether a shift which is flow equivalent with a Property (A) shift is itself a Property (A) shift with the same associated semigroup.

10. Eventual conjugacy

It remains a major open problem to determine whether conjugacy is decidable for shifts of finite type. To attack this problem, a relation called *eventual conjugacy*, also known as *shift equivalence*, was introduced, which may be defined as follows (see [52, Chapter 7] for historical background and other details around this notion). Let n be a positive integer. Recall that A^n denotes the subset of A^+ of words with length n. Considering the natural embedding of $(A^n)^+$ into A^+, one defines the n^{th} *higher power* of a subshift $\mathscr{X}$ of $A^{\mathbb{Z}}$ as the subshift $\mathscr{X}^n$ of $(A^n)^{\mathbb{Z}}$ such that $L(\mathscr{X}^n) = L(\mathscr{X}) \cap (A^n)^+$. Two shifts $\mathscr{X}$ and $\mathscr{Y}$ are *eventually conjugate* if and only if $\mathscr{X}^n$ and $\mathscr{Y}^n$ are conjugate for all sufficiently large n (cf. [52, Definition 1.4.4]). Kim and Roush proved that eventual conjugacy for sofic shifts is decidable [41], but the algorithm which is available is quite intricate. Another deep result by Kim and Roush [42, 43] is that, for shifts of finite type, eventual conjugacy is not the same as conjugacy.

It is easy to check that, for every shift $\mathscr{X} \subseteq A^{\mathbb{Z}}$ and $u, v \in (A^n)^+$, we have $\delta_{\mathscr{X}^n}(u) = \delta_{\mathscr{X}^n}(v)$ if and only if $\delta_{\mathscr{X}}(u) = \delta_{\mathscr{X}}(v)$, and so $S(\mathscr{X}^n)$ embeds naturally in $S(\mathscr{X})$. For sofic shifts, we have the following sort of converse.

Lemma 10.1 ([17, Lemma 5.1])**.** *Let $\mathscr{X}$ be a sofic shift. For each idempotent $e \in S(\mathscr{X})$, choose $u_e \in A^+$ such that $\delta_{\mathscr{X}}(u_e) = e$. Let $A_{\mathscr{X}} = \prod_{e\in E} |u_e|$. Then, for every $n \geq 1$, we have $LU(\mathscr{X}) = LU(\mathscr{X}^{nA_{\mathscr{X}}+1})$.*

Remark 10.2. The proof of Lemma 10.1 depends only on $S(\mathscr{X})$ having a finite set of idempotents. Lemma 10.1 shall be used to extract eventual conjugacy invariants. Since the motivation to study eventual conjugacy is basically restricted to sofic shits, the restriction of our attention in this section to this class of shifts is not very demanding.

Let N be such that $\mathscr{X}^n$ and $\mathscr{Y}^n$ are conjugate for all $n \geq N$. Consider the integer $k = NA_{\mathscr{X}}A_{\mathscr{Y}}+1$. Then $LU(\mathscr{X}^k) = LU(\mathscr{X})$ and $LU(\mathscr{Y}^k) = LU(\mathscr{Y})$ by Lemma 10.1. Since $\mathscr{X}^k$ and $\mathscr{Y}^k$ are conjugate, this justifies the following result, implicitly used in [17].

Corollary 10.3. *If $\mathscr{X}$ and $\mathscr{Y}$ are eventually conjugate sofic shifts, then, for every integer N, there is $k \geq N$, with g.c.d.$(k, N) = 1$, and $\mathscr{X}^k$ and $\mathscr{Y}^k$ conjugate, such that $LU(\mathscr{X}^k) = LU(\mathscr{X})$ and $LU(\mathscr{Y}^k) = LU(\mathscr{Y})$.* □

Remark 10.4. For a labeled graph $\mathfrak{G}$, let $\mathfrak{G}^n$ be the labeled graph over A^n defined as follows: the vertices are those of $\mathfrak{G}$, and an edge from a vertex p to a vertex q, with label $u \in A^n$, is a path in $\mathfrak{G}$ from p to q with label u.

It is easy to see that, up to isomorphism, the Krieger cover of $\mathscr{X}^n$ is the labeled graph $\mathfrak{K}(\mathscr{X})^n$, and, if $\mathscr{X}$ is synchronizing, the Fischer cover of $\mathscr{X}^n$ is $\mathfrak{F}(\mathscr{X})^n$. Moreover, the action of $S(\mathscr{X}^n)$ on $Q(\mathscr{X}^n)$ is the restriction to $S(\mathscr{X}^n)$ of the action of $S(\mathscr{X})$ on $Q(\mathscr{X})$.

For sofic shifts, we have the following consequence of the invariance under conjugacy of the actions $\mathbb{A}_{\mathscr{X}}$ and $\mathbb{A}^{\mathfrak{F}}_{\mathscr{X}}$.

Theorem 10.5. *Suppose that $\mathscr{X}$ and $\mathscr{Y}$ are eventually conjugate sofic shifts. Then the actions $\mathbb{A}_{\mathscr{X}}$ and $\mathbb{A}_{\mathscr{Y}}$ are equivalent. The same happens with the actions $\mathbb{A}^{\mathfrak{F}}_{\mathscr{X}}$ and $\mathbb{A}^{\mathfrak{F}}_{\mathscr{Y}}$ if $\mathscr{X}$ and $\mathscr{Y}$ are irreducible. In particular, the equivalence class of $\mathbb{K}(\mathscr{X})$ is an eventual conjugacy invariant of sofic shifts.*

Proof. By Corollary 10.3, there is k such that $\mathbb{K}(\mathscr{X}^k) = \mathbb{K}(\mathscr{X})$ and $\mathbb{K}(\mathscr{Y}^k) = \mathbb{K}(\mathscr{Y})$. The result follows then from Theorems 4.4 and 4.6 in view of Remark 10.4. □

We next apply Theorem 10.5 to show that several flow equivalence invariants of sofic shifts, which were obtained in the previous sections, are also invariants of eventual conjugacy. In particular, the invariant in Theorem 8.8 will be adapted to this context, and so we define, for a sofic shift $\mathscr{X}$, the *labeled eventual poset of subsynchronizing subshifts of $\mathscr{X}$* as the labeled poset obtained from $\mathsf{Subs}(\mathscr{X})$ by labeling each element of $\mathsf{Subs}(\mathscr{X})$ with its eventual conjugacy class.

Corollary 10.6. *Suppose that $\mathscr{X}$ and $\mathscr{Y}$ are eventually conjugate sofic shifts. Then:*

(1) *the proper communications graphs of $\mathscr{X}$ and $\mathscr{Y}$ are isomorphic;*
(2) *if W is a pseudovariety of semigroups of the form $\mathsf{W} = \mathcal{L}\mathsf{V}$, or, more generally, of the form $\mathsf{W} = \mathsf{V} * \mathsf{D}$, then $S(\mathscr{X}) \in \mathsf{W}$ if and only if $S(\mathscr{Y}) \in \mathsf{W}$;*
(3) *the labeled preordered sets $\mathfrak{KD}(\mathscr{X})$ and $\mathfrak{KD}(\mathscr{Y})$ are isomorphic; moreover, $\mathfrak{FD}(\mathscr{X})$ and $\mathfrak{FD}(\mathscr{Y})$ are also isomorphic if $\mathscr{X}$ and $\mathscr{Y}$ are irreducible;*
(4) *the labeled eventual posets of subsynchronizing subshifts of $\mathscr{X}$ and $\mathscr{Y}$ are isomorphic.*

Proof. In view of the proof of Corollary 5.5, to show item (1) it suffices to check that the posets $P(\mathscr{X})$ and $P(\mathscr{Y})$ are isomorphic. Since $\mathbb{K}(\mathscr{X})$ and $\mathbb{K}(\mathscr{Y})$ are equivalent by Theorem 10.5, the proof given in Corollary 5.2 also yields that fact.

Similarly, the proofs of Theorems 7.1 and 7.5 also yeld item (2).

Item (3) follows immediately from Theorems 10.5 and 6.2.

It remains to show (4). Let k be an integer as in Corollary 10.3. We already know that the labeled eventual poset of subsynchronizing subshifts of a sofic shift is invariant under conjugacy (by Theorem 8.8, see also comments in the paragraph before it). Hence, it suffices to show that the labeled eventual posets of subsynchronizing subshifts of $\mathscr{X}$ and $\mathscr{X}^k$ are the same.

Note that if e is a magic idempotent for $\mathscr{X}$, then it is also a magic idempotent for $\mathscr{X}^k$. Let $m \in (A^k)^+ \subseteq A^+$ be a magic word for $\mathscr{X}^k$ such that $e = \delta_{\mathscr{X}^k}(m)$. Consider the languages

$$L = \{v \in A^+ \mid \exists x \in A^* : mxv \in L(\mathscr{X})\}$$

and

$$K = \{v \in (A^k)^+ \mid \exists x \in (A^k)^* : mxv \in L(\mathscr{X}^k)\}.$$

The word $m \in A^+$ is also magic for $\mathscr{X}$, as $e = \delta_{\mathscr{X}}(m)$. Therefore, since $\mathbb{K}(\mathscr{X}) = \mathbb{K}(\mathscr{X}^k)$, in view of Corollary 8.6, it suffices to show that $L \cap (A^k)^+ = K$. Clearly $K \subseteq L$. Let $v \in L \cap (A^k)^+$, and let $x \in A^*$ be such that $mxv \in L(\mathscr{X})$. Since $g.c.d.(k, N) = 1$, there are positive integers a, b such that $a|m| + |x| = bk$, thus $m^a x \in (A^k)^+$. Hence $m(m^a x)v \in (A^k)^+$. On the other hand, we have $m(m^a x)v \in L(\mathscr{X})$, since $\delta_{\mathscr{X}}(m)$ is idempotent and $mxv \in L(\mathscr{X})$. Therefore, we have $m(m^a x)v \in L(\mathscr{X}^k)$, thus $v \in K$. □

Items (2) and (3) of Corollary 10.6 are from [18] and [17], respectively.

11. Preliminaries for the proof of Theorem 4.4

In this section we introduce some preliminary tools to be used in the proof of Theorem 4.4, which do not concern directly the syntactic semigroup of a shift. Most of the section is about symbolic dynamics, except for some facts about actions of categories.

11.1. **Some categorical preliminaries.** Recall that a category can be viewed as a graph by considering the objects as vertices and the morphisms as edges. Hence we can talk about graph morphisms between categories.

Definition 11.1. Let C and D be small categories, and let $\mathbb{A}$ and $\mathbb{B}$ be faithful actions of C and D, respectively. Denote by $\mathrm{Obj}(C)$ the set of objects of C. Consider a graph morphism $F\colon C \longrightarrow D$ and a family $\eta = (\eta_c)_{c\in\mathrm{Obj}(C)}$ of functions satisfying the following conditions:

(1) for each $c \in \mathrm{Obj}(C)$, η_c is a bijection from $\mathbb{A}(c)$ to $\mathbb{B}(F(c))$;
(2) if $s \in C(c_1, c_2)$ then $\eta_{c_1}(q \cdot s) = \eta_{c_2}(q) \cdot F(s)$, for every $q \in \mathbb{A}(c_2)$.

We say that (F, η) is a *link* between the actions $\mathbb{A}$ and $\mathbb{B}$.

Property 2 in Definition 11.1 asserts the commutativity of the following diagram.

$$\begin{tikzcd} \mathbb{A}(c_2) \arrow[r, "\eta_{c_2}"] \arrow[d, "\_\cdot s"'] & \mathbb{B}(F(c_2)) \arrow[d, "\_\cdot F(s)"] \\ \mathbb{A}(c_1) \arrow[r, "\eta_{c_1}"'] & \mathbb{B}(F(c_1)). \end{tikzcd}$$

Lemma 11.2. *If (F, η) is a link, then F is a faithful functor and $\eta\colon A \Rightarrow B \circ F$ is a natural isomorphism.*

Proof. Let $s \in C(c_2, c_3)$ and $t \in C(c_1, c_2)$. Then $st \in C(c_1, c_3)$. Let $p \in \mathbb{B}(F(c_3))$. Then there is $q \in \mathbb{A}(c_3)$ such that $p = \eta_{c_3}(q)$. Applying several times Property 2 from Definition 11.1, we obtain

$$p \cdot F(st) = \eta_{c_1}(q \cdot st) = \eta_{c_1}((q \cdot s) \cdot t) = \eta_{c_2}(q \cdot s) \cdot F(t) = (p \cdot F(s)) \cdot F(t).$$

Therefore, we have $p \cdot F(st) = p \cdot (F(s)F(t))$ for every $p \in \mathbb{B}(F(c_3))$. From the faithfulness of $\mathbb{B}$, we deduce $F(st) = F(s)F(t)$, whence F is a functor.

In particular, we may consider the functor $\mathbb{B} \circ F$. In this context, properties 1 and 2 in Definition 11.1 mean that η is a natural isomorphism from $\mathbb{A}$ to $\mathbb{B} \circ F$.

Suppose now that s, t are elements of $C(c_1, c_2)$ such that $F(s) = F(t)$. Let $q \in \mathbb{A}(c_2)$. Since $F(s) = F(t)$, we have $\eta_{c_1}(q \cdot s) = \eta_{c_1}(q \cdot t)$. As η_{c_1} is injective, we have $q \cdot s = q \cdot t$, for every $q \in \mathbb{A}(c_2)$. However, $\mathbb{A}$ is a faithful action and so $s = t$, thereby establishing that F is faithful. □

11.2. **Sliding block codes.** A *sliding block code* Φ between subshifts $\mathscr{X}$ of $A^{\mathbb{Z}}$ and $\mathscr{Y}$ of $B^{\mathbb{Z}}$ is a function $\Phi \colon \mathscr{X} \longrightarrow \mathscr{Y}$ for which there are integers $k, l \geq 0$ and a mapping from $A^{k+l+1} \cap L(\mathscr{X})$ to B such that

$$\Phi(x) = (\phi(x_{[i-k,i+l]}))_{i \in \mathbb{Z}}, \quad \text{for all } x \in \mathscr{X}.$$

It will be convenient to extend ϕ to some function $A^{k+l+1} \longrightarrow B$. This extension will also be denoted by ϕ, since it will not be important to know which extension we are choosing. We say that $\phi \colon A^{k+l+1} \longrightarrow B$ is a *block map* of Φ, with *memory* k and *anticipation* l. To describe these data, we use the notation $\Phi = \phi^{[-k,l]} \colon \mathscr{X} \longrightarrow \mathscr{Y}$.

Note that the sliding block code $\Phi \colon \mathscr{X} \longrightarrow \mathscr{Y}$ extends to a sliding block code $A^{\mathbb{Z}} \longrightarrow B^{\mathbb{Z}}$, which we shall also denote by Φ, having $\phi \colon A^{k+l+1} \longrightarrow B$ as block map with memory k and anticipation l. This extension depends only on how is done the extension of the domain of ϕ, from $A^{k+l+1} \cap L(\mathscr{X})$ to A^{k+l+1}.

If $n \geq l$, $m \geq k$ and $\psi \colon A^{m+n+1} \longrightarrow B$ is defined by

$$\psi(a_{-m}a_{-m+1} \cdots a_{n-1}a_n) = \phi(a_{-k}a_{-k+1} \cdots a_{l-1}a_l),$$

with $a_i \in A$, then ψ is a block map of Φ with memory m and anticipation n. In particular, one can choose a block map with equal memory and anticipation.

The following is a well-known and very useful result in symbolic dynamics.

Theorem 11.3 ([34]). *The sliding block codes are precisely the morphisms between shifts.*

We shall say that a sliding block code with memory and anticipation zero is a 1-*block code*. A conjugacy which is a 1-block code is a 1-*conjugacy*. These special types of sliding block codes are of special interest, because of the next lemma.

Lemma 11.4 (cf. [52, Proposition 1.5.12])**.** *Let $\Phi\colon \mathscr{X} \longrightarrow \mathscr{Y}$ be a sliding block code with a block map ϕ having memory and anticipation k. Then there are 1-block codes $\Phi_1\colon \mathscr{Z} \longrightarrow \mathscr{X}$ and $\Phi_2\colon \mathscr{Z} \longrightarrow \mathscr{Y}$ with block maps ϕ_1 and ϕ_2 (respectively) such that Φ_1 is a conjugacy, $\Phi = \Phi_2 \circ \Phi_1^{-1}$ (cf. Diagram (11.1)), and Φ_1^{-1} has a block map ζ with memory and anticipation k such that $\phi = \phi_2 \circ \zeta$.*

$$\begin{tikzcd} & \mathscr{Z} \arrow[dl, "\Phi_1"'] \arrow[dr, "\Phi_2"] & \\ \mathscr{X} \arrow[rr, "\Phi"'] & & \mathscr{Y} \end{tikzcd} \tag{11.1}$$

For the reader's convenience, we give below a proof of Lemma 11.4 (which is formulated in a slightly different way than in [52, Proposition 1.5.12]). The proof is based in the following construction. Let N be a positive integer and define $\beta_N\colon A^{\mathbb{Z}} \longrightarrow (A^N)^{\mathbb{Z}}$ by

$$\beta_N((x_i)_{i\in\mathbb{Z}}) = ([x_{[i,i+N-1]}])_{i\in\mathbb{Z}}.$$

Given a subset $\mathscr{X}$ of $A^{\mathbb{Z}}$, denote by $\mathscr{X}^{[N]}$ the set $\beta_N(\mathscr{X})$. If $\mathscr{X}$ is a subshift of $A^{\mathbb{Z}}$, then $\mathscr{X}^{[N]}$ is a subshift of $(A^N)^{\mathbb{Z}}$ and the map $\beta_N\colon \mathscr{X} \longrightarrow \mathscr{X}^{[N]}$ is a conjugacy [52, Example 1.5.10]. Notice that β_N is given by a block map with memory 0 and anticipation $N-1$.

Proof of Lemma 11.4. We are considering $\mathscr{X}$ and $\mathscr{Y}$ as subshifts of $A^{\mathbb{Z}}$ and $B^{\mathbb{Z}}$, respectively.

Let $C = A^{2k+1}$, and let $\mathscr{Z}$ be the subshift $\mathscr{X}^{[2k+1]}$ of $C^{\mathbb{Z}}$. Let Φ_1 be the inverse of the conjugacy $\beta_{2k+1} \circ \sigma_A^{-k}\colon \mathscr{X} \longrightarrow \mathscr{Z}$. Consider the map $\phi_1\colon C \longrightarrow A$ defined by sending an element w of A^{2k+1} to its middle letter; so if $w = uav$, with $a \in A$ and $|u| = |v| = k$, then $\phi_1(w) = a$. Since

$$\beta_{2k+1} \circ \sigma_A^{-k}((x_i)_{i\in\mathbb{Z}}) = ([x_{[i-k,i+k]}])_{i\in\mathbb{Z}},$$

the map ϕ_1 is a block map for Φ_1 with memory and anticipation zero.

Let $\Phi_2\colon \mathscr{Z} \longrightarrow \mathcal{Y}$ be the sliding block code having the map $\phi\colon C \longrightarrow B$ as a block map with memory and anticipation zero. Then $\Phi_2 \circ \Phi_1^{-1} = \Phi_2 \circ (\beta_{2k+1} \circ \sigma_A^{-k}) = \Phi$.

Finally, note that the identity 1_C on $C = A^{2k+1}$ can be viewed as a block map for $\beta_{2k+1} \circ \sigma_A^{-k}$ with memory and anticipation k. Therefore, we may take $\phi_2 = \phi$ and $\zeta = 1_C$. $\square$

Lemma 11.4 allows us to deduce the existence of a conjugacy invariant by looking only at 1-conjugacies. This strategy will be used in the proof of Theorem 4.4. Indeed, since flow equivalence is the least equivalence relation between shifts containing the conjugacy and symbol expansion relations, Lemma 11.4 permits us to register the following fact.

Lemma 11.5. *Flow equivalence is the least equivalence relation between shifts containing all pairs $(\mathscr{X},\mathscr{Y})$ such that there is either a 1-conjugacy $\mathscr{X}\longrightarrow\mathscr{Y}$ or a symbol expansion $\mathscr{X}\longrightarrow\mathscr{Y}$.*

11.3. **Application of block maps to finite and infinite words.** For alphabets A and B, and a positive integer N, let ϕ be a map from A^N to B. We define a map $\bar{\phi}\colon A^*\longrightarrow B^*$ such that if u is a word of length less than N then $\bar{\phi}(u)=1$, and if u is a word of length at least N, with a factorization $u=u_1\cdots u_m$ such that $u_i\in A$ for all $i\in\{1,\dots,m\}$, then

$$\bar{\phi}(u)=\phi(u_{[1,N]})\phi(u_{[2,N+1]})\cdots\phi(u_{[m-N,m-1]})\phi(u_{[m-N+1,m]}),$$

where $u_{[i,j]}=u_iu_{i+1}\cdots u_{j-1}u_j$ $(1\leq i\leq j\leq m)$. Note that $|\bar{\phi}(u)|=|u|-N+1$ if $|u|\geq N$.

Given a non-negative integer k, denote by t_k the function taking a word $u\in A^*$ to its suffix of length k if $|u|\geq k$; if $|u|<k$, we put $\mathsf{t}_k(u)=u$. Replacing "suffix" by "prefix", we obtain the mapping denoted i_k. Note that $\mathsf{i}_0(u)=\mathsf{t}_0(u)=1$.

Remark 11.6. For all $u,v\in A^*$ we have the following equalities:

$$\bar{\phi}(uv)=\bar{\phi}(u)\cdot\bar{\phi}(\mathsf{t}_{N-1}(u)v)=\bar{\phi}(u\,\mathsf{i}_{N-1}(v))\cdot\bar{\phi}(v). \tag{11.2}$$

Moreover, if $N=2k+1$ then the equality

$$\bar{\phi}(uv)=\bar{\phi}(u\,\mathsf{i}_k(v))\cdot\bar{\phi}(\mathsf{t}_k(u)v), \tag{11.3}$$

also holds, see for instance [18, Lemma 2.5].

Remark 11.7. Consider sliding block codes $\Phi=\phi^{[-k,k]}\colon\mathscr{X}\longrightarrow\mathscr{Y}$ and $\Psi=\psi^{[-l,l]}\colon\mathscr{Y}\longrightarrow\mathscr{Z}$, with $\mathscr{X}\subseteq A^{\mathbb{Z}}$, $\mathscr{Y}\subseteq B^{\mathbb{Z}}$ and $\mathscr{Z}\subseteq C^{\mathbb{Z}}$. Then the map $\zeta\colon A^{2k+2l+1}\longrightarrow C$ such that $\zeta(u)=\psi\circ\bar{\phi}(u)$ is a block map for $\Psi\circ\Phi$ with memory and anticipation $k+l$, and $\bar{\zeta}=\bar{\psi}\circ\bar{\phi}$.

Given alphabets A and B, a positive integer N and a map $\phi\colon A^N\longrightarrow B$, we extend $\bar{\phi}$ to right-infinite and left-infinite words as follows. Define, for $x\in A^{\mathbb{Z}^-}$ and $y\in A^{\mathbb{N}}$, elements $\bar{\phi}(x)\in B^{\mathbb{Z}^-}$ and $\bar{\phi}(x)\in B^{\mathbb{N}}$ by:

$$\bar{\phi}(x)=\cdots\phi(x_{[-(N+2),-3]})\,\phi(x_{[-(N+1),-2]})\,\phi(x_{[-N,-1]}),$$
$$\bar{\phi}(y)=\phi(y_{[0,N-1]})\,\phi(y_{[1,N]})\,\phi(y_{[2,N+1]})\cdots.$$

For $x\in A^{\mathbb{Z}}$, the sequences $\cdots x_{i-2}x_{i-1}x_i$ and $x_ix_{i+1}x_{i+2}\cdots$ are respectively denoted by $x_{]-\infty,i]}$ and $x_{[i,+\infty,[}$.

Remark 11.8. Let $\Phi=\phi^{[-k,k]}\colon A^{\mathbb{Z}}\longrightarrow B^{\mathbb{Z}}$ be a sliding block code. If $x\in A^{\mathbb{Z}}$ then $\Phi(x)=\bar{\phi}(x_{]-\infty,-1+k]}).\bar{\phi}(x_{[i-k,+\infty[})$, for every $i\in\mathbb{Z}$.

Next we list a couple of simple facts about a sliding block code of the form $\Phi\colon\mathscr{X}\longrightarrow\mathscr{Y}$ with block map ϕ, which we shall use without reference:

(1) $\bar{\phi}(L(\mathscr{X}))\subseteq L(\mathscr{Y})\cup\{1\}$.
(2) $\bar{\phi}(\mathscr{X}^-)\subseteq\mathscr{X}^-$ and $\bar{\phi}(\mathscr{X}^+)\subseteq\mathscr{X}^+$.

For a subshift $\mathscr{X}$ of $A^{\mathbb{Z}}$, we denote by $\mathcal{M}_N(\mathscr{X})$ the set of elements of A^+ all of whose factors of length less than or equal to N belong to $L(\mathscr{X})$. The set $\mathcal{M}_N(\mathscr{X})$ is factorial and prolongable, and so it defines a subshift $\mathscr{X}_N$ such that $L(\mathscr{X}_N) = \mathcal{M}_N(\mathscr{X})$. A shift $\mathscr{X}$ is of finite type if and only if $\mathscr{X} = \mathscr{X}_N$ for some $N \geq 1$ (cf. [52, Proposition 2.1.7]).

Lemma 11.9. *Let $\Phi = \phi^{[-k,k]}\colon \mathscr{X} \longrightarrow \mathscr{Y}$ be a conjugacy and let $\Phi^{-1} = \psi^{[-l,l]}\colon \mathscr{Y} \longrightarrow \mathscr{X}$ be its inverse. Take $x \in A^{\mathbb{Z}}$, where $\mathscr{X} \subseteq A^{\mathbb{Z}}$. For every $i \in \mathbb{Z}$, we have*

$$x_{]-\infty,i+k+l]} \in \mathscr{X}^-_{2k+2l+1} \implies \bar{\psi}\bar{\phi}(x_{]-\infty,i+k+l]}) = x_{]-\infty,i]},$$

and

$$x_{[i-(k+l),+\infty[} \in \mathscr{X}^+_{2k+2l+1} \implies \bar{\psi}\bar{\phi}(x_{[i-(k+l),+\infty[}) = x_{[i,+\infty[}.$$

Proof. It suffices to consider the case $x_{]-\infty,i+k+l]} \in \mathscr{X}^-_{2k+2l+1}$, the other case being entirely similar. Replacing x by $\sigma^i(x)$ if necessary, we may assume that $i = 0$.

Let $z = \bar{\phi}(x_{]-\infty,k+l]})$. Then z is the element of $B^{\mathbb{Z}^-}$ given by

$$z_j = \phi(x_{[l-k+j+1,l+k+j+1]}), \quad \text{for all } j \leq -1. \tag{11.4}$$

On the other hand, we have

$$\bar{\psi}(z) = \cdots \psi(z_{[-(2l+1)-2,-3]})\,\psi(z_{[-(2l+1)-1,-2]})\,\psi(z_{[-(2l+1),-1]}). \tag{11.5}$$

From (11.4) we obtain, for all $j \leq 0$, the following equality:

$$\psi(z_{[-(2l+1)+j,-1+j]}) = \psi\bar{\phi}(x_{[j-(k+l),j+k+l]}). \tag{11.6}$$

For every $j \leq 0$, the word $w_j = x_{[j-(k+l),j+k+l]}$ of length $2k+2l+1$ belongs to $L(\mathscr{X})$. According to Remark 11.7, the mapping $\psi\bar{\phi}\colon A^{2k+2l+1} \longrightarrow A$ is a block map for the identity on $\mathscr{X}$ and so, $\psi\bar{\phi}(w_j) = x_j$. Therefore, from (11.5) and (11.6) we obtain $\bar{\psi}\bar{\phi}(x_{]-\infty,k+l]}) = x_{]-\infty,0]}$. □

We conclude this section with a couple of easy lemmas, respectively about infinite and finite words, which are related with Lemma 11.9.

Lemma 11.10. *Let $\Phi = \phi^{[-k,k]}\colon \mathscr{X} \longrightarrow \mathscr{Y}$ be a conjugacy whose inverse has a block map with memory and anticipation l. Suppose that $\mathscr{X} \subseteq A^{\mathbb{Z}}$ and let $x \in A^{\mathbb{Z}}$. Then the following properties hold:*

1. *If we have $x_{]-\infty,i+k+l]} \in \mathscr{X}^-_{2k+2l+1}$ and $\bar{\phi}(x_{]-\infty,i+k+l]}) \in \mathscr{Y}^-$, then we have $x_{]-\infty,i]} \in \mathscr{X}^-$.*
2. *If we have $x_{[i-(k+l),+\infty[} \in \mathscr{X}^+_{2k+2l+1}$ and $\bar{\phi}(x_{[i-(k+l),+\infty[}) \in \mathscr{Y}^+$, then we have $x_{[i,+\infty[} \in \mathscr{X}^+$*
3. *Let $\widehat{\Phi}$ be the sliding block code $A^{\mathbb{Z}} \longrightarrow B^{\mathbb{Z}}$ extending Φ, having $\phi\colon A^{2k+1} \longrightarrow B$ as a block map. If we have $x \in \mathscr{X}_{2k+2l+1}$ and $\widehat{\Phi}(x) \in \mathscr{Y}$, then we have $x \in \mathscr{X}$.*

Proof. We first show (1). Let ψ be the block map such that $\Phi^{-1} = \psi^{[-l,l]}$. Applying Lemma 11.9 to $x_{]-\infty,i+k+l]} \in \mathscr{X}^{-}_{2k+2l+1}$, we obtain $x_{]-\infty,i]} = \bar{\psi}\bar{\phi}(x_{]-\infty,i+k+l]})$. Therefore, since $\bar{\psi}(\mathscr{Y}^-) \subseteq \mathscr{X}^-$, from the hypothesis $\bar{\phi}(x_{]-\infty,i+k+l]}) \in \mathscr{Y}^-$ we get $x_{]-\infty,i]} \in \mathscr{X}^-$. The proof of item (2) is similar.

Finally, suppose that $x \in \mathscr{X}_{2k+2l+1}$ and $\widehat{\Phi}(x) \in \mathscr{Y}$. Let $\widehat{\Phi^{-1}} = \psi^{[-l,l]}$ be the sliding block code $B^{\mathbb{Z}} \longrightarrow A^{\mathbb{Z}}$ extending Φ^{-1}. Then, by Remarks 11.7 and 11.8, we have

$$\widehat{\Phi^{-1}}(\widehat{\Phi}(x)) = \bar{\psi}\bar{\phi}\big(x_{]-\infty,-1-(k+l)]}\big).\bar{\psi}\bar{\phi}\big(x_{[-(k+l),+\infty[}\big).$$

Hence $\widehat{\Phi^{-1}}(\widehat{\Phi}(x)) = x$, by Lemma 11.9. Since $\widehat{\Phi^{-1}}(x) \in \mathscr{Y}$ and the restriction of $\widehat{\Phi^{-1}}$ to $\mathscr{Y}$ is Φ^{-1}, from $\widehat{\Phi}(x) \in \mathscr{Y}$ we conclude that $x \in \mathscr{X}$. □

Lemma 11.11. *Let $\Phi = \phi^{[-k,k]}\colon \mathscr{X} \longrightarrow \mathscr{Y}$ be a conjugacy, and let $\Phi^{-1} = \psi^{[-l,l]}\colon \mathscr{Y} \longrightarrow \mathscr{X}$ be its inverse. Consider an element v of A^+, where $\mathscr{X} \subseteq A^{\mathbb{Z}}$. If r and s are words of length $k+l$ such that $rvs \in \mathcal{M}_{2k+2l+1}(\mathscr{X})$ then $v = \bar{\psi}\bar{\phi}(rvs)$; in particular, if $\bar{\phi}(rvs) \in L(\mathscr{Y})$ then $v \in L(\mathscr{X})$.*

Proof. Since $rvs \in \mathcal{M}_{2k+2l+1}(\mathscr{X})$, there is $x \in \mathscr{X}_{2k+2l+1}$ such that $rvs = x_{[-(k+l),m+k+l]}$, where $m = |v| - 1$. By Lemma 11.9, and since $|s| = k+l$, we have

$$\bar{\psi}\bar{\phi}(x_{]-\infty,m+k+l]}) = x_{]-\infty,m]} = x_{]-\infty,-1]}v.$$

On the other hand, the right infinite sequence $\bar{\psi}\bar{\phi}(x_{]-\infty,m+k+l]})$ ends in the word $\bar{\psi}\bar{\phi}(rvs)$, with length $|rvs| - 2(k+l) = |v|$. Therefore $\bar{\psi}\bar{\phi}(rvs) = v$. □

12. Proof of Theorem 4.4

12.1. **Interplay between block maps and the syntactic congruence.** The content of the next lemma is very similar to that of [17, Proposition 4.2], a result about sofic shifts only. Throughout this section, $\mathscr{X}$ will be a subshift of $A^{\mathbb{Z}}$ and $\mathscr{Y}$ is a subshift of $B^{\mathbb{Z}}$. If w is a word and $w = uv$, then we put $u^{-1}w = v$ and $wv^{-1} = u$.

Lemma 12.1. *Let $\Phi = \phi^{[-l,l]}\colon \mathscr{X} \longrightarrow \mathscr{Y}$ be a conjugacy. Suppose Φ^{-1} has memory and anticipation k. Let $u, v \in A^+$ be words of length greater or equal than $2l$ such that*

$$\mathsf{i}_{2l}(u) = \mathsf{i}_{2l}(v), \quad \mathsf{t}_{2l}(u) = \mathsf{t}_{2l}(v). \tag{12.1}$$

Let p, q be words such that $puq \in L(\mathscr{X})$ and $|p| = |q| = 2k + 2l$. If $\delta_{\mathscr{X}}(u) \subseteq \delta_{\mathscr{X}}(v)$, then $\delta_{\mathscr{Y}}\bar{\phi}(p'uq') \subseteq \delta_{\mathscr{Y}}\bar{\phi}(p'vq')$ whenever $p' \in A^*p$ and $q' \in qA^*$.

Proof. First, we want to prove the following inclusion:

$$\delta_{\mathscr{Y}}\bar{\phi}(puq) \subseteq \delta_{\mathscr{Y}}\bar{\phi}(pvq). \tag{12.2}$$

Let ψ be a block map such that $\Phi^{-1} = \psi^{[-k,k]}$. Let $(x, y) \in \delta_{\mathscr{Y}}\bar{\phi}(puq)$. Since $L(\mathscr{Y})$ is prolongable, there are words x' and y' of length $k+l$ such that $x'x\bar{\phi}(puq)yy'$ belongs to $L(\mathscr{Y})$. Let $z \in \{u, v\}$. The words $\bar{\phi}(p)$ and $\bar{\phi}(q)$ have length $2k$, whence they are respectively the prefix and the suffix

of length $2k$ of $\bar{\phi}(pzq)$, by formula (11.2) in Remark 11.6. Therefore, also by formula (11.2) in Remark 11.6, we have

$$\bar{\psi}(x'x\bar{\phi}(pzq)yy') = \bar{\psi}(x'x\bar{\phi}(p)) \cdot \bar{\psi}\bar{\phi}(pzq) \cdot \bar{\psi}(\bar{\phi}(q)yy') \tag{12.3}$$

The word puq belongs to $L(\mathscr{X})$, and so does pvq because $\delta_{\mathscr{X}}(u) \subseteq \delta_{\mathscr{X}}(v)$. By Lemma 11.11, we then have for each $z \in \{u, v\}$ the equality

$$\bar{\psi}\bar{\phi}(pzq) = \mathsf{i}_{k+l}(p)^{-1}p \cdot z \cdot q\,\mathsf{t}_{k+l}(q)^{-1}. \tag{12.4}$$

For each $z \in \{u, v\}$, denote the word

$$\bar{\psi}(x'x\bar{\phi}(p)) \cdot \mathsf{i}_{k+l}(p)^{-1}p \cdot z \cdot q\,\mathsf{t}_{k+l}(q)^{-1} \cdot \bar{\psi}(\bar{\phi}(q)yy').$$

by w_z. Putting equality (12.3) together with equality (12.4), we get

$$\bar{\psi}(x'x\bar{\phi}(pzq)yy') = w_z. \tag{12.5}$$

Since $x'x\bar{\phi}(puq)yy' \in L(\mathscr{Y})$, we know that $w_u \in L(\mathscr{X})$. Hence, since $\delta_{\mathscr{X}}(u) \subseteq \delta_{\mathscr{X}}(v)$, we also have $w_v \in L(\mathscr{X})$. By formula (11.2) in Remark 11.6, we have

$$x'x\bar{\phi}(pvq)yy' = x'x \cdot \bar{\phi}(p\,\mathsf{i}_{2l}(v)) \cdot \bar{\phi}(v) \cdot \bar{\phi}(\mathsf{t}_{2l}(v)q) \cdot yy'.$$

The words $\bar{\phi}(p\,\mathsf{i}_{2l}(v))$ and $\bar{\phi}(\mathsf{t}_{2l}(v)q)$ have length $2k + 2l$, therefore every factor of length $2k + 2l + 1$ of $x'x\bar{\phi}(pvq)yy'$ is a factor of at least one of the words

$$x'x\phi(p\,\mathsf{i}_{2l}(v)), \quad \bar{\phi}(p\,\mathsf{i}_{2l}(v)) \cdot \bar{\phi}(v) \cdot \bar{\phi}(\mathsf{t}_{2l}(v)q), \quad \bar{\phi}(\mathsf{t}_{2l}(v)q) \cdot yy'.$$

By (12.1), the first and the last of these three words belong to $L(\mathscr{Y})$, since they are factors of $x'x\bar{\phi}(puq)yy'$, itself an element of $L(\mathscr{Y})$. The second word also belongs to $L(\mathscr{Y})$, since it equals $\bar{\phi}(pvq)$ and $pvq \in L(\mathscr{X})$. Therefore $x'x\bar{\phi}(pvq)yy'$ belongs to $\mathcal{M}_{2k+2l+1}(\mathscr{Y})$. Then, as $w_v \in L(\mathscr{X})$, from (12.5) and Lemma 11.11 we deduce that $x\bar{\phi}(pvq)y \in L(\mathscr{X})$. This concludes the proof of the inclusion (12.2).

Using formula (11.2) in Remark 11.6, we obtain

$$\begin{aligned}\bar{\phi}(p'zq') &= \bar{\phi}(p'p^{-1} \cdot pzq \cdot q^{-1}q') \\ &= \bar{\phi}(p'p^{-1} \cdot \mathsf{i}_{2l}(p)) \cdot \bar{\phi}(pzq) \cdot \bar{\phi}(\mathsf{t}_{2l}(q) \cdot q^{-1}q').\end{aligned}$$

Hence, multiplying both sides in (12.2) on the left by $\delta_{\mathscr{Y}}\bar{\phi}(p'p^{-1} \cdot \mathsf{i}_{2l}(p))$ and on the right by $\delta_{\mathscr{Y}}\bar{\phi}(\mathsf{t}_{2l}(q) \cdot q^{-1}q')$, we get $\delta_{\mathscr{Y}}\bar{\phi}(p'uq') \subseteq \delta_{\mathscr{Y}}\bar{\phi}(p'vq')$. □

The following definition and subsequent technical lemmas will be later used to build an equivalence $\mathbb{K}(\mathscr{X}) \longrightarrow \mathbb{K}(\mathscr{Y})$ starting from an 1-conjugacy $\mathscr{X} \longrightarrow \mathscr{Y}$.

Definition 12.2. Let $\Phi = \phi^{[-l,l]}\colon \mathscr{X} \longrightarrow \mathscr{Y}$ be a conjugacy. Let e be an idempotent of $S(\mathscr{X}) \setminus \{0\}$. For each positive integer N, define the set $W_{\phi,N}(e)$ as the set of words $w \in A^+$ such that $|w| > N$, $\delta_{\mathscr{X}}(w) = e$ and $\delta_{\mathscr{Y}}\bar{\phi}(\mathsf{t}_l(w)\,w^n\,\mathsf{i}_l(w))$ is an idempotent of $S(\mathscr{Y}) \setminus \{0\}$ for every positive integer n.

Lemma 12.3. *The set $W_{\phi,N}(e)$ is non-empty.*

Proof. Let u be a word such that $\delta_{\mathscr{X}}(u) = e$. Suppose that Φ^{-1} has memory and anticipation k. Let $v = u^{\max\{N,2l,l+2k\}}$. Then $\delta_{\mathscr{X}}(v^n) = e$ for every $n \geq 1$. Since $e \neq 0$, the word v^n belongs to $L(\mathscr{X})$. Its length is at least $2l$. Note also that $\mathsf{t}_l(v)\, v$ and $v\, \mathsf{i}_l(v)$ have length greater than or equal to $l + (l + 2k) = 2k + 2l$. Then, for every $n \geq 3$, we may apply Lemma 12.1 to the inclusions $\delta_{\mathscr{X}}(v^{n-2}) \subseteq \delta_{\mathscr{X}}(v^{2n-2})$ and $\delta_{\mathscr{X}}(v^{2n-2}) \subseteq \delta_{\mathscr{X}}(v^{n-2})$, with the words $\mathsf{t}_l(v)\, v$ and $v\, \mathsf{i}_l(v)$ playing the role of p' and q', respectively. We deduce that $\delta_{\mathscr{Y}}\bar{\phi}(\mathsf{t}_l(v)\, v^n\, \mathsf{i}_l(v)) = \delta_{\mathscr{Y}}\bar{\phi}(\mathsf{t}_l(v)\, v^{2n}\, \mathsf{i}_l(v))$ for every $n \geq 3$. Since we have $\bar{\phi}(\mathsf{t}_l(v)\, v^m\, \mathsf{i}_l(v)) = \bar{\phi}(\mathsf{t}_l(v)\, v\, \mathsf{i}_l(v))^m$ for every m (cf., (11.3) of Remark 11.6), it follows that $\delta_{\mathscr{Y}}\bar{\phi}(\mathsf{t}_l(v)\, v^n\, \mathsf{i}_l(v))$ is idempotent whenever $n \geq 3$. This idempotent is not 0, because $v^m \in L(\mathscr{X})$ and so $\bar{\phi}(v^m) \in L(\mathscr{Y})$, for all $m \geq 1$. Therefore $v^3 \in W_{\phi,N}(e)$. $\square$

Lemma 12.4. *Consider a* 1*-conjugacy* $\Phi = \phi^{[0,0]}\colon \mathscr{X} \longrightarrow \mathscr{Y}$ *whose inverse has memory and anticipation* k. *Let* $e, f \in S(\mathscr{X})\setminus\{0\}$. *Given* $w_e \in W_{\phi,2k}(e)$ *and* $w_f \in W_{\phi,2k}(f)$, *let* u, v *be elements of* $w_e A^+ w_f \cap L(\mathscr{X})$. *Then* $\delta_{\mathscr{X}}(u) \subseteq \delta_{\mathscr{X}}(v)$ *if and only if* $\delta_{\mathscr{Y}}\bar{\phi}(u) \subseteq \delta_{\mathscr{Y}}\bar{\phi}(v)$.

Proof. Suppose that $\delta_{\mathscr{X}}(u) \subseteq \delta_{\mathscr{X}}(v)$. Note that $\delta_{\mathscr{X}}(u) = \delta_{\mathscr{X}}(w_e u w_f)$, because $u \in w_e A^+ w_f$ and $\delta_{\mathscr{X}}(w_e)$, $\delta_{\mathscr{X}}(w_f)$ are idempotents. Therefore, from $u \in L(\mathscr{X})$ we get $w_e u w_f \in L(\mathscr{X})$. As $|w_e|, |w_f| > 2k$, from Lemma 12.1 we deduce $\delta_{\mathscr{Y}}\bar{\phi}(w_e u w_f) \subseteq \delta_{\mathscr{Y}}\bar{\phi}(w_e v w_f)$. Since $u, v \in w_e A^+ w_f$ and $\delta_{\mathscr{Y}}\bar{\phi}(w_e)$, $\delta_{\mathscr{Y}}\bar{\phi}(w_f)$ are idempotents, we in fact have $\delta_{\mathscr{Y}}\bar{\phi}(u) \subseteq \delta_{\mathscr{Y}}\bar{\phi}(v)$.

Conversely, suppose that $\delta_{\mathscr{Y}}\bar{\phi}(u) \subseteq \delta_{\mathscr{Y}}\bar{\phi}(v)$. Let ψ be a block map of Φ^{-1} with memory and anticipation k. Note that $\mathsf{i}_{2k}(\bar{\phi}(u)) = \mathsf{i}_{2k}(\bar{\phi}(v)) = \mathsf{i}_{2k}(\bar{\phi}(w_e))$ and $\mathsf{t}_{2k}(\bar{\phi}(u)) = \mathsf{t}_{2k}(\bar{\phi}(v)) = \mathsf{t}_{2k}(\bar{\phi}(w_f))$. Since $w_e u w_f \in L(\mathscr{X})$, we have $\bar{\phi}(\mathsf{t}_{2k}(w_e) u\, \mathsf{i}_{2k}(w_f)) \in L(\mathscr{Y})$. Then, as $|\bar{\phi}(w_e)| = |w_e| > 2k$ and $|\bar{\phi}(w_f)| = |w_f| > 2k$, it follows from Lemma 12.1 that

$$\delta_{\mathscr{X}}\bar{\psi}\bar{\phi}(\mathsf{t}_{2k}(w_e) u\, \mathsf{i}_{2k}(w_f)) \subseteq \delta_{\mathscr{X}}\bar{\psi}\bar{\phi}(\mathsf{t}_{2k}(w_e) v\, \mathsf{i}_{2k}(w_f)). \tag{12.6}$$

We have already observed that $w_e u w_f \in L(\mathscr{X})$. Using the same arguments, we get $w_e v w_f \in L(\mathscr{X})$. We may then apply Lemma 11.11 to deduce, for each $z \in \{u, v\}$, the equality $\bar{\psi}\bar{\phi}(\mathsf{t}_{2k}(w_e) z\, \mathsf{i}_{2k}(w_f)) = \mathsf{t}_k(w_e) z\, \mathsf{i}_k(w_f)$. Therefore (12.6) translates to

$$\delta_{\mathscr{X}}(\mathsf{t}_k(w_e) u\, \mathsf{i}_k(w_f)) \subseteq \delta_{\mathscr{X}}(\mathsf{t}_k(w_e) v\, \mathsf{i}_k(w_f)). \tag{12.7}$$

If we multiply both sides of (12.7) on the left by $\delta_{\mathscr{X}}(w_e \cdot (\mathsf{t}_k(w_e))^{-1})$ and on the right by $\delta_{\mathscr{X}}((\mathsf{i}_k(w_f))^{-1} \cdot w_f)$, we obtain $\delta_{\mathscr{X}}(w_e u w_f) \subseteq \delta_{\mathscr{X}}(w_e v w_f)$, that is, $\delta_{\mathscr{X}}(u) \subseteq \delta_{\mathscr{X}}(v)$. $\square$

12.2. **Invariance of the action $\mathbb{A}_{\mathscr{X}}$ under 1-conjugacies.** The following definition introduces our standing notation.

Definition 12.5. Let $\Phi = \phi^{[0,0]}\colon \mathscr{X} \longrightarrow \mathscr{Y}$ be a 1-conjugacy. We define a graph morphism $F_\Phi\colon \mathbb{K}(\mathscr{X}) \longrightarrow \mathbb{K}(\mathscr{Y})$ as follows. If $\Phi^{-1} = \psi^{[-k,k]}$ has memory and anticipation k, choose for each $e \in E(S(\mathscr{X})) \setminus \{0\}$, an element

w_e of $W_{\phi,2k}(e)$. For each morphism (e,s,f) of $\mathbb{K}(\mathscr{X})$ such that $s \neq 0$, let $W_{(e,s,f)}$ be the non-empty set $w_e A^+ w_f \cap \delta_{\mathscr{X}}^{-1}(s)$. Then we define:

(1) $F_\Phi(0) = 0$;
(2) $F_\Phi(e) = \delta_{\mathscr{Y}}\bar{\phi}(w_e)$ for every object $e \in E(S(\mathscr{X})) \setminus \{0\}$;
(3) $F_\Phi(e,0,f) = (F_\Phi(e), 0, F_\Phi(f))$ for all objects e, f of $\mathbb{K}(\mathscr{X})$;
(4) for every morphism (e,s,f) with $s \in S(\mathscr{X}) \setminus \{0\}$ put

$$F_\Phi(e,s,f) = \big(F_\Phi(e), \delta_{\mathscr{Y}}\bar{\phi}(u), F_\Phi(f)\big)$$

where $u \in W_{(e,s,f)}$ (this is well defined by Lemma 12.4).

We shall see later that F_Φ is an equivalence of categories.

The following lemma is a step towards the construction of an isomorphism $A_{\mathscr{X}} \Rightarrow A_{\mathscr{Y}} \circ F_\Phi$. The setup for the lemma is that of Definition 12.5.

Lemma 12.6. *Let $e \in S(\mathscr{X}) \setminus \{0\}$ and $x \in A^{\mathbb{Z}}$. Suppose that $xw_e \in \mathscr{X}^-$ and $yw_e \in \mathscr{X}^-_{2k+1}$. Then we have the following equivalence:*

$$C_{\mathscr{X}}(xw_e) \subseteq C_{\mathscr{X}}(yw_e) \iff C_{\mathscr{Y}}(\bar{\phi}(xw_e)) \subseteq C_{\mathscr{Y}}(\bar{\phi}(yw_e)).$$

Proof. Suppose $C_{\mathscr{X}}(xw_e) \subseteq C_{\mathscr{X}}(yw_e)$. Let $z \in C_{\mathscr{Y}}(\bar{\phi}(xw_e))$. Because $\delta_{\mathscr{Y}}\bar{\phi}(w_e)$ is idempotent, we have $C_{\mathscr{Y}}(\bar{\phi}(xw_e)) = C_{\mathscr{Y}}(\bar{\phi}(xw_ew_e))$, and so

$$\bar{\phi}(xw_ew_e).z \in \mathscr{Y}.$$

Therefore,

$$\bar{\psi}\bar{\phi}[xw_e\mathsf{i}_k(w_e)].\bar{\psi}\big[\bar{\phi}(\mathsf{t}_k(w_e)w_e)z\big] \in \mathscr{X} \tag{12.8}$$

by Remark 11.6 Since $xw_e \in \mathscr{X}^-$ and $\delta_{\mathscr{X}}(w_e)$ is idempotent, we also have $xw_ew_e \in \mathscr{X}^-$. Therefore, $\bar{\psi}\bar{\phi}[xw_e\mathsf{i}_k(w_e)] = xw_e$ by Lemma 11.9. From (12.8) and the hypothesis $C_{\mathscr{X}}(xw_e) \subseteq C_{\mathscr{X}}(yw_e)$, we obtain

$$yw_e.\bar{\psi}\big[\bar{\phi}(\mathsf{t}_k(w_e)w_e)z\big] \in \mathscr{X}.$$

Hence

$$\bar{\phi}(yw_e).\bar{\phi}\bar{\psi}\big[\bar{\phi}(\mathsf{t}_k(w_e)w_e)z\big] \in \mathscr{Y}. \tag{12.9}$$

As $\bar{\phi}(w_ew_e)z \in \mathscr{Y}^+$, from Lemma 11.9 we get $\bar{\phi}\bar{\psi}\big[\bar{\phi}(\mathsf{t}_k(w_e)w_e)z\big] = \bar{\phi}(w_e)z$. Therefore (12.9) is the same as $z \in C_{\mathscr{Y}}(\bar{\phi}(yw_ew_e))$. Since $C_{\mathscr{Y}}(\bar{\phi}(yw_e)) = C_{\mathscr{Y}}(\bar{\phi}(yw_ew_e))$, this proves $C_{\mathscr{Y}}(\bar{\phi}(xw_e)) \subseteq C_{\mathscr{Y}}(\bar{\phi}(yw_e))$.

Conversely, suppose $C_{\mathscr{Y}}(\bar{\phi}(xw_e)) \subseteq C_{\mathscr{Y}}(\bar{\phi}(yw_e))$. Let $z \in C_{\mathscr{X}}(xw_e)$. Then $xw_e.z \in \mathscr{X}$ and $\bar{\phi}(xw_e).\bar{\phi}(z) = \Phi(xw_e.z) \in \mathscr{Y}$. By the hypothesis, we get $\bar{\phi}(yw_e).\bar{\phi}(z) \in \mathscr{Y}$. Since $yw_e \in \mathscr{X}^-_{2k+1}$, $w_ez \in \mathscr{X}^+$ and $|w_e| \geq 2k$, we know that $yw_e.z \in \mathscr{X}_{2k+1}$. Applying Lemma 11.10(3), we obtain $yw_e.z \in \mathscr{X}$. This proves $C_{\mathscr{X}}(xw_e) \subseteq C_{\mathscr{X}}(yw_e)$. □

Definition 12.7. Let e be an idempotent of $S(\mathscr{X}) \setminus \{0\}$. Note that every element of $Q(\mathscr{X})e$ is of the form $C_{\mathscr{X}}(xw_e)$, where $x \in A^{\mathbb{Z}^-}$. We define a mapping

$$\eta_e \colon Q(\mathscr{X})e \longrightarrow Q(\mathscr{Y})F_\Phi(e)$$

by $\eta_e(\emptyset) = \emptyset$ and $\eta_e(C_{\mathscr{X}}(xw_e)) = C_{\mathscr{Y}}(\bar{\phi}(xw_e))$ whenever $C_{\mathscr{X}}(xw_e) \neq \emptyset$. Note that η_e is a well-defined and injective mapping by Lemma 12.6. If $e = 0$, then η_e is defined as the unique mapping $\{\emptyset\} \longrightarrow \{\emptyset\}$.

Lemma 12.8. *For each idempotent e of $S(\mathscr{X})$, the mapping η_e is bijective.*

Proof. We assume $e \neq 0$. Since η_e is injective and $\eta_e(\emptyset) = \emptyset$, it remains to show that the set $Q(\mathscr{Y})F_\Phi(e) \setminus \{\emptyset\}$ is contained in the image of η_e.

An element of $Q(\mathscr{Y})F_\Phi(e) \setminus \{\emptyset\}$ is of the form $C_{\mathscr{Y}}(y\bar{\phi}(w_e))$, for some $y \in B^{\mathbb{Z}^-}$ such that $y\bar{\phi}(w_e) \in \mathscr{Y}^-$. For such an element, let $y' = y\bar{\phi}(w_e\, \mathsf{i}_k(w_e))$. As $F_\Phi(e) = \delta_{\mathscr{Y}}\bar{\phi}(w_e)$ is idempotent, we have

$$C_{\mathscr{Y}}(y\bar{\phi}(w_e)) = C_{\mathscr{Y}}(y\bar{\phi}(w_e^n)),$$

for all $n \geq 2$, which implies that $y' \in \mathscr{Y}^-$, since $C_{\mathscr{Y}}(y\bar{\phi}(w_e))$ is non-empty.

Let $x = \bar{\psi}(y')$. By Lemma 11.9, we have $\bar{\phi}(x) = y\bar{\phi}(w_e)$, whence

$$C_{\mathscr{Y}}(y\bar{\phi}(w_e)) = C_{\mathscr{Y}}(\bar{\phi}(x)). \tag{12.10}$$

We claim that $xw_e \in \mathscr{X}^-$. Since $C_{\mathscr{Y}}(y\bar{\phi}(w_e^3)) \neq \emptyset$, we know that $y'' = y\bar{\phi}(w_e^2\, \mathsf{i}_k(w_e))$ belongs to $\mathscr{X}^-$. Therefore $x'' = \bar{\psi}(y'')$ belongs to $\mathscr{X}$. Note that $x'' = x\bar{\psi}\bar{\phi}[\mathsf{t}_k(w_e)w_e\, \mathsf{i}_k(w_e)]$. Hence $x'' = xw_e$ by Lemma 11.11, which establishes the claim.

Since $xw_e \in \mathscr{X}^-$, we have $\eta_e(C_{\mathscr{X}}(xw_e)) = C_{\mathscr{Y}}(\bar{\phi}(xw_e))$. Therefore, from equalities (12.10) and $C_{\mathscr{Y}}(y\bar{\phi}(w_e)) = C_{\mathscr{Y}}(y\bar{\phi}(w_e^2))$, we obtain the equality $\eta_e(C_{\mathscr{X}}(xw_e)) = C_{\mathscr{Y}}(y\bar{\phi}(w_e))$. This concludes the proof that η_e is onto. □

We wish to show that F_Φ is a functor and that η is a natural isomorphism $A_{\mathscr{X}} \longrightarrow A_{\mathscr{Y}} \circ F_\Phi$. This is done in the following proposition, via Lemma 11.2. We can apply Lemma 11.2 because $\mathbb{A}_{\mathscr{X}}$ is faithful (Remark 4.3).

Proposition 12.9. *Suppose there is a 1-conjugacy $\Phi\colon \mathscr{X} \longrightarrow \mathscr{Y}$. Take all data from Definitions 12.7 and 12.5. The pair (F_Φ, η) is a link between the faithful actions $\mathbb{A}_{\mathscr{X}}$ and $\mathbb{A}_{\mathscr{Y}}$, and F_Φ is an equivalence.*

Proof. Throughout the proof, F_Φ is denoted simply by F. Otherwise, we retain the notation and assumptions of Definition 12.7.

We begin by showing that if $q \in Q(\mathscr{X})e$, then

$$\eta_f(q \cdot (e, s, f)) = \eta_e(q) \cdot F(e, s, f). \tag{12.11}$$

The cases in which $q = \emptyset$ or $s = 0$ are immediate, since $\eta_e(\emptyset) = \eta_f(\emptyset) = \emptyset$ and $F(e, 0, f) = (F(e), 0, F(f))$. Suppose that $q \neq \emptyset$ and $s \neq 0$. We may take an element $x \in A^{\mathbb{Z}^-}$ such that $xw_e \in \mathscr{X}^-$ and $q = C_{\mathscr{X}}(xw_e)$, and an element $u \in W_{(e,s,f)}$. Then

$$q \cdot (e, s, f) = C_{\mathscr{X}}(xw_e) \cdot \delta_{\mathscr{X}}(u) = C_{\mathscr{X}}(xw_eu)$$

and

$$\eta_e(q) \cdot F(e, s, f) = C_{\mathscr{Y}}(\bar{\phi}(xw_e)) \cdot \delta_{\mathscr{Y}}(\bar{\phi}(u)) = C_{\mathscr{Y}}(\bar{\phi}(xw_euw_f)),$$

where the last equality holds because $\delta_{\mathscr{Y}}(\bar{\phi}(u)) = \delta_{\mathscr{Y}}(\bar{\phi}(uw_f))$ by Lemma 12.4. Suppose $\eta_e(q) \cdot F(e, s, f) \neq \emptyset$. Then we have $\bar{\phi}(xw_euw_fw_f) \in \mathscr{Y}^-$. Since $xw_e \in \mathscr{X}^-$, $w_euw_fw_f \in L(\mathscr{X})$ and $|w_e| > 2k$, we also have $xw_euw_fw_f \in$

$\mathscr{X}^-_{2k+1}$. We then get $xw_euw_f\mathsf{t}_k(w_f)^{-1} \in \mathscr{X}^-$ by Lemma 11.10(1). In particular, $C_{\mathscr{X}}(xw_eu) \neq \emptyset$. We have therefore proved that $q \cdot (e,s,f) = \emptyset$ implies $\eta_e(q) \cdot F(e,s,f) = \emptyset$. In the case $q \cdot (e,s,f) \neq \emptyset$, we have

$$\eta_f(q \cdot (e,s,f)) = C_{\mathscr{Y}}(\bar{\phi}(xw_eu)) = \eta_e(q) \cdot F(e,s,f),$$

which establishes (12.11) in all cases.

Therefore, by Lemma 11.2, we know that F is a faithful functor and that η is a natural isomorphism $\mathbb{A}_{\mathscr{X}} \Rightarrow \mathbb{A}_{\mathscr{Y}} \circ F$.

Next, we show that F is essentially surjective. Let f be an idempotent of $S(\mathscr{Y}) \setminus \{0\}$. By Lemma 12.3, there is $w \in B^+$ such that $|w| > 2k$, $\delta_{\mathscr{Y}}(w) = f$ and $e = \delta_{\mathscr{X}}\bar{\psi}(\mathsf{t}_k(w)\, w\, \mathsf{i}_k(w))$ is an idempotent of $S(\mathscr{X}) \setminus \{0\}$. Let $v = \bar{\psi}(\mathsf{t}_k(w)\, w\, \mathsf{i}_k(w))$. We have $e = \delta_{\mathscr{X}}(w_e) = \delta_{\mathscr{X}}(v)$. Since e is idempotent and $e \neq 0$, we deduce that $(v^*w_e^*)^* \setminus \{1\} \subseteq L(\mathscr{X})$. We may then apply Lemma 12.1 to the equality $\delta_{\mathscr{X}}(w_e) = \delta_{\mathscr{X}}(v)$, first with $p' = q' = v$, and second with $p' = q' = w_e$, to deduce that

$$\delta_{\mathscr{Y}}\bar{\phi}(vw_ev) = \delta_{\mathscr{Y}}\bar{\phi}(v^3) \quad \text{and} \quad \delta_{\mathscr{Y}}\bar{\phi}(w_e^3) = \delta_{\mathscr{Y}}\bar{\phi}(w_evw_e). \tag{12.12}$$

Note that $\bar{\phi}(v) = w$ by Lemma 11.11. Therefore, what we have in (12.12) is $fF(e)f = f$ and $F(e) = F(e)fF(e)$, thus $f \mathrel{\mathcal{R}} fF(e) \mathrel{\mathcal{L}} F(e)$. By Lemma 2.3, this implies that f and $F(e)$ are isomorphic objects of $\mathbb{K}(\mathscr{Y})$.

It remains to show that F is full. Let e and f be idempotents of $S(\mathscr{X}) \setminus \{0\}$. Let μ be a morphism of $\mathbb{K}(\mathscr{Y})$ from $F(f)$ to $F(e)$. We want to show that $\mu = F(\nu)$ for some morphism ν of $\mathbb{K}(\mathscr{X})$ from f to e. Since $F(e,0,f) = (F(e), 0, F(f))$, we may suppose that $\mu = (F(e), \delta_{\mathscr{Y}}(v), F(f))$ for some $v \in L(\mathscr{Y})$. Let v' be the word

$$\bar{\phi}(\mathsf{t}_k(w_e)\, w_e^2)\, v\, \bar{\phi}(w_f^2\, \mathsf{i}_k(w_f)).$$

Note that v' is a factor of $v'' = \bar{\phi}(w_e)^3\, v\, \bar{\phi}(w_f)^3$, and because μ is a morphism of $\mathbb{K}(\mathscr{Y})$, we have $\delta_{\mathscr{Y}}(v'') = \delta_{\mathscr{Y}}(v)$. Since $v \in L(\mathscr{Y})$, we conclude that v'' and v' belong to $L(\mathscr{Y})$. Hence $\bar{\psi}(v') \in L(\mathscr{X})$. Let $z_e = \bar{\phi}(\mathsf{t}_k(w_e)\, w_e\, \mathsf{i}_k(w_e))$ and $z_f = \bar{\phi}(\mathsf{t}_k(w_f)\, w_f\, \mathsf{i}_k(w_f))$. Observe that $\bar{\psi}(v') \in \bar{\psi}(z_e)A^+\bar{\psi}(z_f)$. Note also that $\mathsf{t}_k(w_e)\, w_e\, \mathsf{i}_k(w_e)$ and $\mathsf{t}_k(w_f)\, w_f\, \mathsf{i}_k(w_f)$ belong to $L(\mathscr{X})$, thus $\bar{\psi}(z_e) = w_e$ and $\bar{\psi}(z_f) = w_f$ by Lemma 11.11. Hence, if $s = \delta_{\mathscr{X}}\bar{\psi}(v')$ then (e,s,f) is a morphism of $\mathbb{K}(\mathscr{X})$ and $\bar{\psi}(v') \in W_{(e,s,f)}$. Then

$$F(e,s,f) = (F(e), \delta_{\mathscr{Y}}\bar{\phi}\bar{\psi}(v'), F(f)).$$

Since the words $\bar{\phi}(\mathsf{t}_k(w_e))$ and $\bar{\phi}(\mathsf{i}_k(w_f))$ have length k, by Lemma 11.11 we have $\bar{\phi}\bar{\psi}(v') = \bar{\phi}(w_e)^2\, v\, \bar{\phi}(w_f)^2$, hence

$$F(e,s,f) = (F(e), F(e)\, \delta_{\mathscr{Y}}(v)\, F(f), F(f)) = \mu,$$

thus showing F is full. $\square$

12.3. **Invariance of the action $\mathbb{A}_{\mathscr{X}}$ under symbol expansion.** The reader should review the definitions and notation from Subsection 3.6.

Remark 12.10. Using induction on the length of words, one verifies that

$$\mathcal{E}(A^*) = B^* \setminus \Big(\diamond B^* \cup B^*\alpha \cup \bigcup_{x\in A\setminus\{\diamond\}} B^*\alpha x B^* \cup \bigcup_{x\in A\setminus\{\alpha\}} B^* x \diamond B^*\Big).$$

Remark 12.10 justifies several simple and useful facts, like the following.

Lemma 12.11. *Let $v \in A^+$. For $x, y, u \in B^*$, if $x\mathcal{E}(v)y = \mathcal{E}(u)$ then $x, y \in \mathcal{E}(A^*)$ and $u = \mathcal{E}^{-1}(x)v\mathcal{E}^{-1}(y)$. Consequently, if $\mathcal{E}(v) \in L(\mathscr{X}')$ then $v \in L(\mathscr{X})$.*

Proof. The first part of the lemma follows from Remark 12.10 and the fact that $\mathcal{E}$ is injective. If $\mathcal{E}(v) \in L(\mathscr{X}')$ then there is $u \in L(\mathscr{X})$ and $x, y \in B^*$ with $\mathcal{E}(u) = x\,\mathcal{E}(v)\,y$. From $u = \mathcal{E}^{-1}(x)\,v\,\mathcal{E}^{-1}(y)$, we deduce $v \in L(\mathscr{X})$. □

A direct consequence of Lemma 12.11 is the following analog.

Lemma 12.12. *Let $x \in A^{\mathbb{Z}}$. If $\mathcal{E}(x) \in \mathscr{X}'$ then $x \in \mathscr{X}$.*

Proof. If u is a finite block of x, then $\mathcal{E}(u) \in L(\mathscr{X}')$. By Lemma 12.11, we have $u \in L(\mathscr{X})$. Hence $x \in \mathscr{X}$. □

In this subsection we prove the invariance of the action $\mathbb{A}_{\mathscr{X}}$ under symbol expansion following a script which differs slightly from the proof of the invariance under 1-conjugacy given in Subsection 12.2. The difference stems from the next proposition, in which we obtain a special homomorphism from $S(\mathscr{X})$ to $S(\mathscr{X}')$, called a local isomorphism, a result with some independent interest. This proposition immediately yields an equivalence $F\colon \mathbb{K}(\mathscr{X}) \longrightarrow \mathbb{K}(\mathscr{X}')$. The construction of a natural isomorphism $\mathbb{A}_{\mathscr{X}} \Rightarrow \mathbb{A}_{\mathscr{X}'} \circ F$ will come later.

In [51], a semigroup homomorphism $\theta\colon S \longrightarrow T$ is termed a *local isomorphism* if the following conditions are satisfied:

1. (1) $\theta|_{eSf}$ is a bijection of eSf with $\theta(e)T\theta(f)$;
2. (2) if $e' \in E(\theta(S))$, then there is an idempotent $e \in S$ with $\theta(e) = e'$;
3. (3) for each idempotent $e \in T$, there is an idempotent $f \in \theta(S)$ with $e\,\mathcal{D}\,f$.

(Actually, Lawson only defines the notion for semigroups with local units.) It is immediate from the definition (cf. [51]) that if $\theta\colon S \longrightarrow T$ is an isomorphism, then θ induces an equivalence $\Theta\colon \mathbb{K}(S) \longrightarrow \mathbb{K}(T)$ given by $\Theta(e) = \theta(e)$ on objects and $\Theta(e, s, f) = (\theta(e), \theta(s), \theta(f))$ on morphisms.

Proposition 12.13. *There is a well-defined homomorphism $\mathcal{E}'\colon S(\mathscr{X}) \longrightarrow S(\mathscr{X}')$ sending $\delta_{\mathscr{X}}(u)$ to $\delta_{\mathscr{X}'}(\mathcal{E}(u))$ and 0 to 0. Moreover, $\mathcal{E}'$ is a local isomorphism.*

Proof. We begin by showing that $\delta_{\mathscr{X}}(u) \subseteq \delta_{\mathscr{X}}(v)$ if and only if $\delta_{\mathscr{X}'}(\mathcal{E}(u)) \subseteq \delta_{\mathscr{X}'}(\mathcal{E}(v))$ for $u, v \in A^+$. Suppose that $\delta_{\mathscr{X}}(u) \subseteq \delta_{\mathscr{X}}(v)$ and let x and y be words such that $x\,\mathcal{E}(u)\,y$ belongs to $L(\mathscr{X}')$. There are words x' and y' such that $x'x\,\mathcal{E}(u)\,yy'$ belongs to $L(\mathscr{X}') \cap \mathcal{E}(A^+)$. By Lemma 12.11, we have $\mathcal{E}^{-1}(x'x)\,u\,\mathcal{E}^{-1}(y'y) \in L(\mathscr{X})$. Since $\delta_{\mathscr{X}}(u) \subseteq \delta_{\mathscr{X}}(v)$, it follows that the

word $z = \mathcal{E}^{-1}(x'x)\, v\, \mathcal{E}^{-1}(y'y)$ also belongs to $L(\mathscr{X})$. Hence $x\, \mathcal{E}(v)\, y$ belongs to $L(\mathscr{X}')$, since it is a factor of $\mathcal{E}(z)$. Therefore $\delta_{\mathscr{X}'}\mathcal{E}(u) \subseteq \delta_{\mathscr{X}'}\mathcal{E}(v)$. Let $z \in A^+ \setminus L(\mathscr{X})$. Then $\mathcal{E}(z) \notin L(\mathscr{X}')$, again by Lemma 12.11. Therefore, we have $\mathcal{E}'(\delta_{\mathscr{X}}(z)) = 0 = \mathcal{E}'(0)$.

This proves $\mathcal{E}'$ is a well-defined homomorphism.

On the other hand, if $\delta_{\mathscr{X}'}\mathcal{E}(u) \subseteq \delta_{\mathscr{X}'}\mathcal{E}(v)$ then, for every $x, y \in A^*$, we have the following chain of implications, where the last one uses Lemma 12.11:

$$xuy \in L(\mathscr{X}) \Rightarrow \mathcal{E}(xuy) \in L(\mathscr{X}') \Rightarrow \mathcal{E}(xvy) \in L(\mathscr{X}') \Rightarrow xvy \in L(\mathscr{X}).$$

This proves that $\delta_{\mathscr{X}}(u) \subseteq \delta_{\mathscr{X}}(v)$. It follows that $\mathcal{E}'$ is injective.

The second condition in the definition of a local isomorphism is clearly satisfied by $\mathcal{E}'$ because it is injective.

Suppose that $f = \delta_{\mathscr{X}'}(w)$ is an idempotent of $S(\mathscr{X}')$ with $f \notin \mathcal{E}'(S(\mathscr{X}))$. Then $w \notin \mathcal{E}(A^+)$ on the one hand, and $w \in L(\mathscr{X}')$, on the other hand (the latter because $f \neq 0$). Since $w \in L(\mathscr{X}')$, there is $v \in L(\mathscr{X})$ such that $\mathcal{E}(v) = pwq$ for some p, q. Let u be a (possibly empty) word of maximal length such that $\mathcal{E}(u)$ is a factor of w, and let a, b be words such that $w = a\, \mathcal{E}(u)\, b$. Since $\mathcal{E}(v) = pa\, \mathcal{E}(u)\, bq$, it follows that $pa, bq \in \mathcal{E}(A^+)$ by Lemma 12.11. By the maximality of u, we have $a, b \in \{1, \alpha, \diamond\}$. Note also that $\{a, b\} \neq \{1\}$, because $w \notin \mathcal{E}(A^+)$. Since $\delta_{\mathscr{X}'}(w)$ is idempotent, the word $w^2 = a\, \mathcal{E}(u)\, ba\, \mathcal{E}(u)\, b$ belongs to $L(\mathscr{X}')$, thus $ra\, \mathcal{E}(u)\, ba\, \mathcal{E}(u)\, bs \in \mathcal{E}(A^+)$ for some words r, s. Then ba belongs to $\mathcal{E}(A^+)$ by Remark 12.10. The only possibility is $ba = \alpha\diamond$, thus $w = \diamond\, \mathcal{E}(u)\, \alpha$. Then, we have:

$$\delta_{\mathscr{X}'}\mathcal{E}(\alpha u)^2 = \delta_{\mathscr{X}'}(\alpha \diamond \mathcal{E}(u)\, \alpha \diamond \mathcal{E}(u)) = \delta_{\mathscr{X}'}(\alpha w \diamond \mathcal{E}(u)).$$

On the other hand, since $\delta_{\mathscr{X}'}(w)$ is idempotent, we also have:

$$\delta_{\mathscr{X}'}(\alpha w \diamond \mathcal{E}(u)) = \delta_{\mathscr{X}'}(\alpha w^3 \diamond \mathcal{E}(u)) = \delta_{\mathscr{X}'}((\alpha \diamond \mathcal{E}(u))^4) = \delta_{\mathscr{X}'}\mathcal{E}(\alpha u)^4.$$

Therefore, $\delta_{\mathscr{X}'}\mathcal{E}(\alpha u)^2$ is an idempotent in the image of $\mathcal{E}'$. Denote this idempotent by e. Let $x = \delta_{\mathscr{X}'}(\alpha w)$ and $y = \delta_{\mathscr{X}'}(w^2 \diamond \mathcal{E}(u))$. Then $xy = e$ and $yx = f$, thus $e\, \mathcal{D}\, f$.

It remains to show that $\mathcal{E}'(eS(\mathscr{X})f) = \mathcal{E}'(e)S(\mathscr{X}')\mathcal{E}'(f)$, whenever $e, f \in E(S(\mathscr{X}))$. Let u and v be elements of A^+ such that $e = \delta_{\mathscr{X}}(u)$ and $f = \delta_{\mathscr{X}}(v)$. Clearly, $0 \in \mathcal{E}'(eS(\mathscr{X})f)$. Let $w \in L(\mathscr{X}')$ be such that $\delta_{\mathscr{X}'}(w) \in \mathcal{E}'(e)S(\mathscr{X}')\mathcal{E}'(f)$. Since $\delta_{\mathscr{X}'}(w) = \delta_{\mathscr{X}'}(\mathcal{E}(u)\, w\, \mathcal{E}(v))$, there are words p and q such that $p\, \mathcal{E}(u)\, w\, \mathcal{E}(v)\, q$ belongs to $L(\mathscr{X}') \cap \operatorname{Im} \mathcal{E}$. From Remark 12.10 it follows that $w = \mathcal{E}(w')$ for some $w' \in A^+$. Moreover, $w' \in L(\mathscr{X})$ by Lemma 12.11. Clearly, $\delta_{\mathscr{X}}(uw'v) \in eS(\mathscr{X})f$. Furthermore, $\mathcal{E}'\delta_{\mathscr{X}}(uw'v) = \mathcal{E}'(e)\, \delta_{\mathscr{X}'}(w)\, \mathcal{E}'(f) = \delta_{\mathscr{X}'}(w)$, completing the proof. □

Definition 12.14. Consider the mapping $F_{\mathcal{E}} \colon \mathbb{K}(\mathscr{X}) \longrightarrow \mathbb{K}(\mathscr{X}')$ defined as follows:

(1) $F_{\mathcal{E}}(e) = \mathcal{E}'(e)$ if e is an object of $\mathbb{K}(\mathscr{X})$;
(2) $F_{\mathcal{E}}(e, s, f) = \big(\mathcal{E}'(e), \mathcal{E}'(s), \mathcal{E}'(f)\big)$ if (e, s, f) is morphism of $\mathbb{K}(\mathscr{X})$.

It follows immediately from Proposition 12.13 and the remarks preceding it that that $F_{\mathcal{E}}$ is an equivalence. We record this here.

Proposition 12.15. *The functor $F_{\mathcal{E}}\colon \mathbb{K}(\mathscr{X}) \longrightarrow \mathbb{K}(\mathscr{X}')$ is an equivalence.*

Remark 12.16. Propositions 12.9 and 12.15 respectively establish the invariance of $\mathbb{K}(\mathscr{X})$, up to natural equivalence, under 1-conjugacies and symbol expansions. By Lemma 11.5, this means that at this point, with the proof of Theorem 4.4 not yet concluded, we have proved the invariance of $\mathbb{K}(\mathscr{X})$, up to natural equivalence, under flow equivalence (Theorem 4.2).

We proceed with the proof of Theorem 4.4, by steps in order to produce a natural isomorphism $\mathbb{A}_{\mathscr{X}} \Rightarrow \mathbb{A}_{\mathscr{X}'} \circ F_{\mathcal{E}}$.

Lemma 12.17. *We have*

$$C_{\mathscr{X}}(x) \subseteq C_{\mathscr{X}}(y) \iff C_{\mathscr{X}'}(\mathcal{E}(x)) \subseteq C_{\mathscr{X}'}(\mathcal{E}(y)),$$

for every $x, y \in A^{\mathbb{Z}^-}$.

Proof. Suppose that $C_{\mathscr{X}'}(\mathcal{E}(x)) \subseteq C_{\mathscr{X}'}(\mathcal{E}(y))$. Let $z \in C_{\mathscr{X}}(x)$, that is $x.z \in \mathscr{X}$. Then $\mathcal{E}(x).\mathcal{E}(z) = \mathcal{E}(x.z) \in \mathscr{X}'$. Hence $\mathcal{E}(y).\mathcal{E}(z) = \mathcal{E}(y.z) \in \mathscr{X}'$, by hypothesis. By Lemma 12.12, we have $y.z \in \mathscr{X}$. We have therefore proved $C_{\mathscr{X}}(x) \subseteq C_{\mathscr{X}}(y)$.

Suppose now that $C_{\mathscr{X}}(x) \subseteq C_{\mathscr{X}}(y)$. Let $z \in C_{\mathscr{X}'}(\mathcal{E}(x))$, that is $\mathcal{E}(x).z \in \mathscr{X}'$. By Remark 12.10, we have $z = \mathcal{E}(t)$ for some unique $t \in A^{\mathbb{N}}$. Moreover, $x.t \in \mathscr{X}$ by Lemma 12.12. Since $C_{\mathscr{X}}(x) \subseteq C_{\mathscr{X}}(y)$, we obtain $y.t \in \mathscr{X}$, whence $\mathcal{E}(y).z \in \mathscr{X}'$. Therefore, $C_{\mathscr{X}'}(\mathcal{E}(x)) \subseteq C_{\mathscr{X}'}(\mathcal{E}(y))$. □

Consider the map $h\colon Q(\mathscr{X}) \longrightarrow Q(\mathscr{X}')$ such that $h(C_{\mathscr{X}}(x)) = C_{\mathscr{X}'}(\mathcal{E}(x))$, for every $x \in A^{\mathbb{Z}^-}$ By Lemma 12.17, this is a well-defined injective function.

Lemma 12.18. *We have $h(q \cdot s) = h(q) \cdot \mathcal{E}'(s)$, for every $q \in Q(\mathscr{X})$ and $s \in S(\mathscr{X})$.*

Proof. Take $u \in A^+$ such that $s = \delta_{\mathscr{X}}(u)$, and let $x \in A^{\mathbb{Z}^-}$ be such that $q = C_{\mathscr{X}}(x)$. Then $q \cdot s = C_{\mathscr{X}}(x)$ and

$$h(q \cdot s) = C_{\mathscr{X}'}(\mathcal{E}(xu)) = C_{\mathscr{X}'}(\mathcal{E}(x)) \cdot \delta_{\mathscr{X}'}(\mathcal{E}(u)).$$

Since $\mathcal{E}'(s) = \delta_{\mathscr{X}'}(\mathcal{E}(u))$, this concludes the proof. □

Lemma 12.19. *The equality $h(Q(\mathscr{X}) \cdot s) = Q(\mathscr{X}') \cdot \mathcal{E}'(s)$ holds for every $s \in S(\mathscr{X})$.*

Proof. By Lemma 12.18, we have $h(Q(\mathscr{X}) \cdot s) \subseteq Q(\mathscr{X}') \cdot \mathcal{E}'(s)$. Conversely, let $q \in Q(\mathscr{X}') \cdot \mathcal{E}'(s)$. We want to show that $q \in h(Q(\mathscr{X}) \cdot s)$. Since $\emptyset = \emptyset \cdot 0$, by Lemma 12.18 we have $h(\emptyset) = h(\emptyset) \cdot 0 = \emptyset$. Therefore, we may suppose $q \neq \emptyset$. Take $u \in A^+$ such that $s = \delta_{\mathscr{X}}(u)$. Then, there is $y \in B^{\mathbb{Z}^-}$ such that $q = C_{\mathscr{X}'}(y\,\mathcal{E}(u))$. The assumption $q \neq 0$ means that $y\,\mathcal{E}(u) \in (\mathscr{X}')^-$, and so by Remark 12.10 there is $\tilde{y} \in B^{\mathbb{Z}^-}$ such that $y = \mathcal{E}(\tilde{y})$. Then $h(C_{\mathscr{X}}(\tilde{y}u)) = q$. Since $C_{\mathscr{X}}(\tilde{y}u) \in Q(\mathscr{X}) \cdot s$, this concludes the proof. □

We are now ready to exhibit a natural isomorphism $\mathbb{A}_{\mathscr{X}} \Rightarrow \mathbb{A}_{\mathscr{X}'} \circ F_{\mathcal{E}}$.

Proposition 12.20. *For each idempotent $e \in S(\mathscr{X})$, let η_e be the function $Q(\mathscr{X})e \longrightarrow Q(\mathscr{X}')\,\mathcal{E}'(e)$ such that $\eta_e(r) = h(r)$ for every $r \in Q(\mathscr{X})e$. Let $\eta = (\eta_e)_{e\in E(S(\mathscr{X}))}$. Then η is a natural isomorphism $\mathbb{A}_{\mathscr{X}} \Rightarrow \mathbb{A}_{\mathscr{X}'} \circ F_{\mathcal{E}}$.*

Proof. By Lemma 12.19, the co-domain of η_e is correctly defined, and η_e is bijective (as h is injective). On the other hand, by Lemma 12.18, the family η is a natural transformation from $\mathbb{A}_{\mathscr{X}}$ to $\mathbb{A}_{\mathscr{X}'} \circ F_{\mathcal{E}}$. □

12.4. **Conclusion.**

Conclusion of the proof of Theorem 4.4. Let θ, ϑ and τ be the binary relations on the class of shift spaces defined as follows:

(1) $\mathscr{X}\ \theta\ \mathscr{Y}$ if and only if there is a 1-conjugacy from $\mathscr{X}$ onto $\mathscr{Y}$;
(2) $\mathscr{X}\ \vartheta\ \mathscr{Y}$ if and only if $\mathscr{Y}$ is a symbol expansion of $\mathscr{X}$;
(3) $\mathscr{X}\ \tau\ \mathscr{Y}$ if $\mathbb{A}_{\mathscr{X}} \sim \mathbb{A}_{\mathscr{Y}}$ (recall Definition 2.4).

Note that Propositions 12.9 and 12.20 entail respectively $\theta \subseteq \tau$ and $\vartheta \subseteq \tau$. By Lemma 11.5, flow equivalence is the least symmetric transitive relation containing θ and ϑ. Hence, to conclude the proof of Theorem 4.4 it suffices to observe that τ is an equivalence relation in the class of shift spaces, since $\sim$ is itself an equivalence relation (Remark 2.5). □

13. Proof of Theorem 4.6

In the proof of the following lemma we apply some results from Section 5 which were proved independently from Theorem 4.6.

Lemma 13.1. *Let $\mathscr{X}$ and $\mathscr{Y}$ by synchronizing shifts. Consider a natural equivalence $F\colon \mathbb{K}(\mathscr{X}) \longrightarrow \mathbb{K}(\mathscr{Y})$. If e is an object of $\mathbb{K}(\mathscr{X})$ then $\eta_e(Q_{\mathfrak{F}}(\mathscr{X})e) = Q_{\mathfrak{F}}(\mathscr{Y})F(e)$.*

Proof. Let $q \in Q_{\mathfrak{F}}(\mathscr{X})e$, where e is an idempotent of $S(\mathscr{X})$. Then $q = qe \in Q_{\mathfrak{F}}(\mathscr{X})$. By Remark 5.3, we have $\eta_e(q) \in Q_{\mathfrak{F}}(\mathscr{Y})$. As noted in Proposition 5.1(1), we have $\eta_e(q)F(e) = \eta_e(q)$, whence $\eta_e(Q_{\mathfrak{F}}(\mathscr{X})e) \subseteq Q_{\mathfrak{F}}(\mathscr{Y})F(e)$. Conversely, suppose that $p \in Q_{\mathfrak{F}}(\mathscr{Y})F(e)$. Then $p = \eta_e(q)$ for some unique $q \in Q(\mathscr{X})e$. Again by Remark 5.3, we have $q \in Q_{\mathfrak{F}}(\mathscr{X})e$. Therefore $\eta_e(Q_{\mathfrak{F}}(\mathscr{X})e) = Q_{\mathfrak{F}}(\mathscr{Y})F(e)$. □

Conclusion of the proof of Theorem 4.6. Let $\mathscr{X}$ and $\mathscr{Y}$ be flow equivalent synchronizing shifts. By Theorem 4.4, there is an equivalence $F\colon \mathbb{K}(\mathscr{X}) \longrightarrow \mathbb{K}(\mathscr{Y})$ for which there is a natural isomorphism $\eta\colon \mathbb{A}_{\mathscr{X}} \longrightarrow \mathbb{A}_{\mathscr{Y}} \circ F$. Hence, the following diagram commutes, for every arrow (e,s,f) of $\mathbb{K}(\mathscr{Y})$:

$$\begin{tikzcd} Q(\mathscr{X})e \arrow[r, "\eta_e"] \arrow[d, "\_\cdot(e,s,f)"'] & Q(\mathscr{Y})F(e) \arrow[d, "\_\cdot F(e,s,f)"] \\ Q(\mathscr{X})f \arrow[r, "\eta_f"] & Q(\mathscr{Y})F(f). \end{tikzcd}$$

By Lemma 13.1, we may consider the bijective function $\eta'_e\colon Q_{\mathfrak{F}}(\mathscr{X})e \longrightarrow Q_{\mathfrak{F}}(\mathscr{Y})F(e)$ obtained by restriction and co-restriction of η_e. Therefore, we have the following commutative diagram:

$$\begin{tikzcd} Q_{\mathfrak{F}}(\mathscr{X})e \arrow[r, "\eta'_e"] \arrow[d, "\_\cdot(e,s,f)"'] & Q_{\mathfrak{F}}(\mathscr{Y})F(e) \arrow[d, "\_\cdot F(e,s,f)"] \\ Q_{\mathfrak{F}}(\mathscr{X})f \arrow[r, "\eta'_f"] & Q_{\mathfrak{F}}(\mathscr{Y})F(f). \end{tikzcd}$$

That is, the family $(\eta'_e)_{e\in E(S(\mathscr{X}))}$ is an isomorphism $\mathbb{A}^{\mathfrak{F}}_{\mathscr{X}} \Rightarrow \mathbb{A}^{\mathfrak{F}}_{\mathscr{Y}} \circ F$. $\square$

Appendix A. A remark on ordered semigroups

Recall that the underlying set of the semigroup $S(\mathscr{X})$ can be viewed as the set of syntactic contexts. The semigroup $S(\mathscr{X})$, was investigated in [18] enriched with this partial order, making it the *ordered syntactic semigroup of* $L(\mathscr{X})$; see [66] for an introductory text on ordered semigroups. At the category level, one considers *ordered categories* as in [67]: briefly, a partial order compatible with composition is defined in each hom-set. The category $\mathbb{K}(\mathscr{X})$ was considered in [18] as an ordered category, with order inherited from that of $S(\mathscr{X})$, that is, one has $(e,s,f)\leq(e,t,f)$ if and only if $s\subseteq t$, whenever (e,s,f) and (e,t,f) are arrows of $\mathbb{K}(\mathscr{X})$. We remark that Theorem 4.2 generalizes to ordered categories (Theorem A.1 below). This generalization demands a definition of what an equivalence of ordered categories is, something not in the literature we know. The notion of functor of ordered categories appears in [67], as well the related notions of full and essentially surjective (which coincide with those used for categories) and faithful (which is more demanding in this new setting: for such a functor F and arrows s and t with the same domain and the same co-domain, one has $s\leq t$ if and only if $F(s)\leq F(t)$). It is an easy exercise to prove that a faithful functor of ordered categories is an equivalence as a functor of (usual) categories if and only if it has a quasi-inverse which is also a faithful functor of ordered categories. This leads to the following definition: two ordered categories C and D are equivalent if there is a faithful functor of ordered categories $C\longrightarrow D$ that is also an equivalence of (usual) categories.

Theorem A.1 ("Ordered" version of Theorem 4.2)**.** *If $\mathscr{X}$ and $\mathscr{Y}$ are flow equivalent shifts, then the ordered categories $\mathbb{K}(\mathscr{X})$ and $\mathbb{K}(\mathscr{Y})$ are equivalent.*

Proof. The functor F_Φ from Definition 12.5 and Proposition 12.9 is a faithful functor of ordered categories, that is, we have $(e,s,f)\leq(e,t,f)$ if and only if $F_\Phi(e,s,f)\leq F_\Phi(e,t,f)$: this is clearly true if $s=0$ because $F_\Phi(e,0,f)=(F_\Phi(e),0,F_\Phi(f))$, and also if $s\neq 0$ (which implies $t\neq 0$) by Lemma 12.4 and the definition of F_Φ. On the other hand, according to what is written at the beginning of Proposition 12.13, we also know that the functor $F_{\mathcal{E}}$ from Definition 12.14 is a faithful functor of ordered categories. This concludes the proof, in view of Lemma 11.5 and Propositions 12.9 and 12.15. $\square$

Theorem A.1 originates more refined invariants than Theorem 4.2, as seen in [18], but Theorem 4.2 seems rich enough, and more simple to handle.

Appendix B. Symbol expansion and subsynchronizing subshifts

Now that we have the tools developed in Subsection 12.3, we are able to show the proposition needed to conclude the proof of Theorem 8.8.

Proposition B.1. *Let $\mathscr{X}$ be a sofic shift, and let e be a magic idempotent of $S(\mathscr{X})$. Consider a symbol expansion $\mathscr{X}'$ of $\mathscr{X}$, defined by a symbol expansion homomorphism $\mathcal{E}$. If e is a magic idempotent for $\mathscr{X}$, then the subsynchronizing subshift $S(F_{\mathcal{E}}(e))$ of $\mathscr{X}'$ is the symbol expansion of $S(e)$ defined by $\mathcal{E}$.*

Proof. Let $u \in L(\mathscr{X})$ be such that $e = \delta_{\mathscr{X}}(u)$.

Consider an element of $L(S(e)')$ of the form $\mathcal{E}(v)$, with $v \in L(S(e))$. Then $uwv \in L(\mathscr{X})$ for some word w, and so $\mathcal{E}(uwv) \in L(\mathscr{X}')$. Since $F_{\mathcal{E}}(e) = \delta_{\mathscr{X}'}\mathcal{E}(u)$, this shows that $\mathcal{E}(v)$ is a finite block of $S(F_{\mathcal{E}}(e))$. Every finite block of $S(e)'$ is a factor of a word such as $\mathcal{E}(v)$. Therefore, we proved that $S(e)' \subseteq S(F_{\mathcal{E}}(e))$.

Conversely, let v be a finite block of $S(F_{\mathcal{E}}(e))$. Then $\mathcal{E}(u)wvw'\mathcal{E}(u') \in L(\mathscr{X}')$ for some words w, w', u', with u' a non-empty word over the alphabet of $\mathscr{X}$. By Remark 12.10, the words $\mathcal{E}(u)wvw'\mathcal{E}(u')$ and wvw' belong to $\mathrm{Im}\mathcal{E}$, and so $u\,\mathcal{E}^{-1}(wvw')u' \in L(\mathscr{X})$. This shows that $\mathcal{E}^{-1}(wvw') \in S(e)$. Therefore, wvw' belongs to $L(S(e)')$, and hence so does its factor v. This shows that $S(F_{\mathcal{E}}(e)) \subseteq S(e)'$. □

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